Part XI

Classifying Frieze Patterns

The goal for this part is to classify frieze patterns.

Observations about frieze patterns

- True or False: A rotation symmetry of a strip must be a $1/2$ turn.
- True or False: If a strip has a mirror reflection symmetry, then the mirror line must be either horizontal or vertical: the mirror cannot be "tilted".
- True or False: If a strip has glide reflection symmetry, then the glide line must be horizontal.
- True or False: If a strip has glide reflection symmetry, then the (smallest) translation distance for a glide (g) must be half of the (smallest) translation distance for a translation (t).

What types of symmetries are possible for frieze patterns?

- Translation: Mandatory or optional? Restrictions?
- Glide reflection: Mandatory or optional? Restrictions?
- Reflection: Mandatory or optional? Restrictions?
- Rotation: Mandatory or optional? Restrictions?

How many ways to combine three symbols?

- We'll use the following letters to represent symmetries in frieze patterns.
  1. H = horizontal mirror
  2. V = vertical mirror
  3. R = rotation
  4. G = glide
- Since translation is required, we won't list it.
- Each frieze pattern combines some or all (or none) of H, V, R, and G.
- How many ways are there to write some or all or none of the letters H, V, R, and G?

Which combinations of the three symbols represent frieze patterns?

H
V
R
G

HV
HR
HG
HR
VR
VG
RG

HVR
HVG
VRG

HVRG
translation
only

For each of these 16 combinations, try drawing a border pattern that has the symmetries listed and only those symmetries.

- Which ones can actually be achieved?
- What rules do you notice when you combine symmetries?

Rules for frieze patterns

H
V
R
G

HV
HR
HG
HR
VR
VG
RG

HVR
HVG
VRG

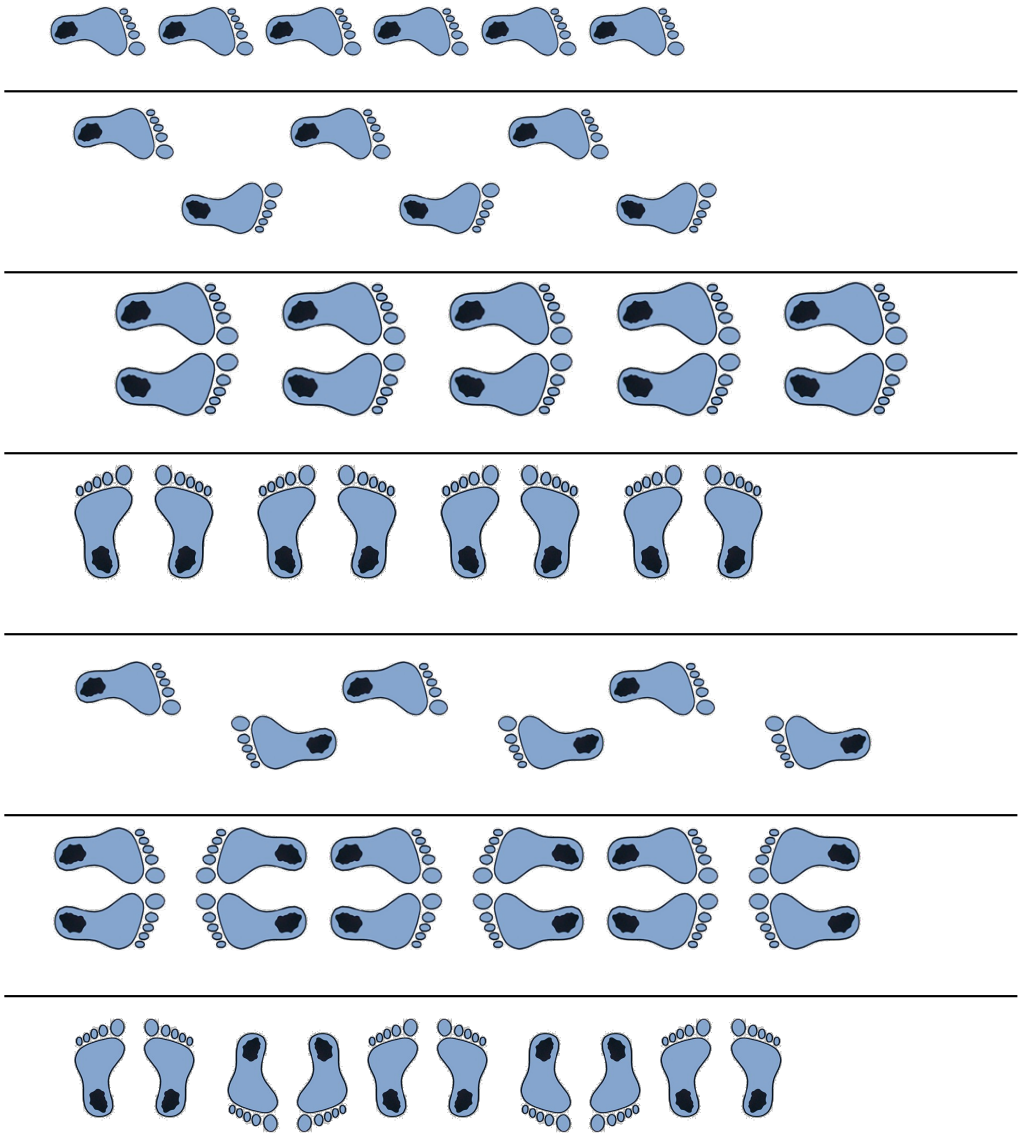
HVRG
translation
only

For example, if we draw a pattern with H and V (using the letter R as the motif), we get:

This pattern is forced to have additional symmetries. Which ones?

Classification of frieze patterns

These are the seven types of frieze patterns.



Technical details

But could there be two different types of frieze patterns that both have V only?
In the pattern we found with V only, there were two different types of vertical mirrors.
Could there be a frieze pattern with V only that has only one type of vertical mirrors? Or three types?

To be sure there are no other types of frieze patterns, we need to consider each of the seven achievable symmetry combinations (like V only, or HVRG) and verify that there is only one way to achieve this combination.

Practice identifying frieze patterns

Identify the following frieze patterns using both the HVRG notation and the hop-step-jump notation.

pppppppp

$p^d p^d p^d p$

pqpqpqp

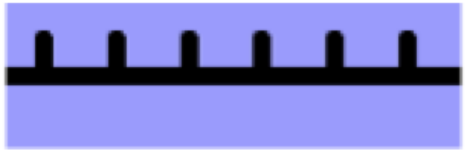
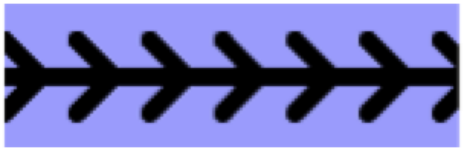
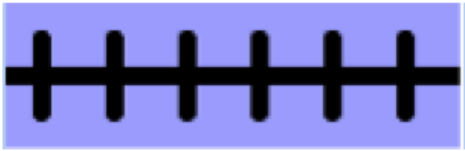
$p^b p^b p^b p$

bbbbbbbbb
pppppppppp

bdbdbdbdb
pqpqpqpqp

bdbdbdbdb
qpqpqpqpq

Alternative notations for frieze patterns

Notations				Description	Examples
IUC	Orbifold	Coxeter	Schönflies*		
p1	∞∞	$[\infty]^+$	C_∞	(hop)	
p1m1	*∞∞	[∞]	$C_{\infty v}$	(sidle)	
p11m	∞*	$[\infty^+,2]$	$C_{\infty h}$	(jump)	
p11g	∞×	$[\infty^+,2^+]$	S_∞	(step)	
p2	22∞	$[2,\infty]^+$	D_∞	(spinning hop)	
p2mg	2*∞	$[2^+,\infty]$	$D_{\infty d}$	(spinning sidle)	
p2mm	*22∞	[2,∞]	$D_{\infty h}$	(spinning jump)	

From Wikipedia.

Crystallographic notation

Crystallographic notation uses a string of four characters based on the following rules:

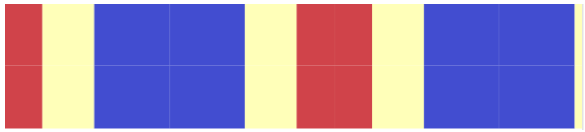
1. The first character is always p.
2. . The second character is:
   - 2 if there is rotational symmetry (of degree 2)
   - 1 if there is no rotational symmetry
3. The third character is:
   - m if there is a vertical mirror
   - 1 if there is no vertical mirror
4. The fourth character is:
   - m if there is a horizontal mirror
   - g if there is a glide but no horizontal mirror
   - 1 otherwise

Sometimes these signatures are shortened by leaving off the trailing 1s.

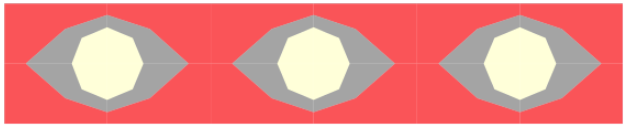
Snakeskin frieze patterns

Identify the following frieze patterns using the hop-step-jump notation and crysallographic notation.

A. Arizona Coral Snake



B. Florida Scarlet Snake



C. Southern Copperhead

D. Spotted Night Snake

E. Trans-Pecos Rat Snake

F. Western Diamondback Rattlesnake

From Jim Wilson, University of Georgia

Homework

1. (Ungraded and optional, but good practice.) Identify the symmetry types of the textbook examples of coffee friezes and Sonny Bono friezes using hop-step-jump or crystallographic notation. The answers are on p. 73 of the textbook in orbifold notation.
2. Identify the symmetry types of these external link: Iroquois and Ojibwa inspired patterns using hop-step-jump or crystallographic notation.
3. Find at least 5 different frieze patterns that are used as architectural features on the inside or outside of campus buildings. They do not all have to have different symmetry types. For each one,
   - 3.1 take a picture (preferably) or draw a sketch,
   - 3.2 record its location (e.g. the name of the building), and
   - 3.3 identify the symmetry type using either hop-step-jump or crystallographic notation.

More homework

4. Briefly explain why the following combinations of symmetries (with no other symmetries present besides translation) are not possible for frieze patterns. Hint: remember the rules we found in class.
   - H
   - HV
   - HR
   - VR
   - VG
   - RG
   - HVR
   - HVG
   - HRG
5. Also (ungraded) please submit an image that has wallpaper symmetry using the Sakai Assignments tab. Wallpaper symmetry means that it has translational symmetry in two directions, not just one direction like our strip patterns.