Part XIX

Euler Characteristic and Topology

The goal for this part is to classify topological surfaces based on their Euler characteristic and orientability.

Euler characteristic of some familiar surfaces

Find the Euler characteristic for:

1. a torus
2. a 2-holed torus
3. a cylinder (without the top or bottom)
4. a cone (without the bottom)
5. a hexagon
6. a Mobius band

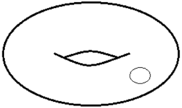
Note that the cylinder, cone, hexagon, and Mobius band are *s*urfaces with boundary.

What numbers are possible as the Euler number for surfaces?

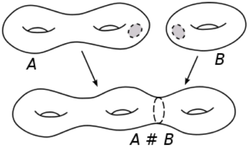
Building new surfaces out of old

Two ways of building new surfaces out of old are:

1. making punctures



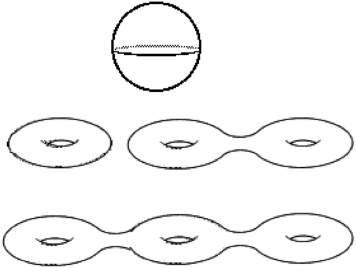
2. joining surfaces with connected sums



In fact, all surfaces can be built this way, starting only the with building blocks of spheres, tori, and projective planes!
See Conway's zip proof in Chapter 8.

Building all orientable surfaces

- Fact: any (finite) orientable surface without boundary is topologically equivalent to either
  1. a sphere
  2. a connected sum of one or more tori

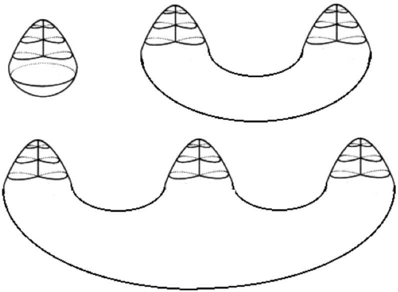


S^2
T^2
$T^2 \# T^2$
$T^2 \# T^2 \# T^2$
$T^2 \# T^2 \# T^2 \# T^2$
$\ldots$

- Any (finite) orientable surface with boundary with topologically equivalent to one of these with surfaces with one or more punctures.
- Is there a similar way to characterize non-orientable surfaces?

Building all non-orientable surfaces

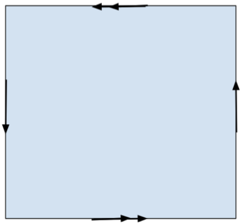
Fact: any (finite) non-orientable surface without boundary is topologically equivalent to a connected sum of one or more projective planes.

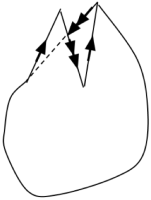


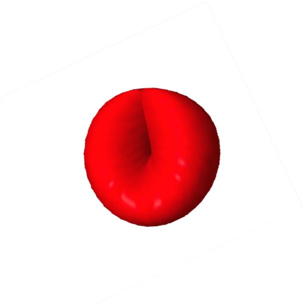
P^2
$P^2 \# P^2$
$P^2 \# P^2 \# P^2$
$P^2 \# P^2 \# P^2 \# P^2$
$\ldots$

Any (finite) non-orientable surface with boundary with topologically equivalent to one of these with surfaces with one or more punctures.

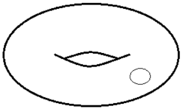
Remember the Projective Plane



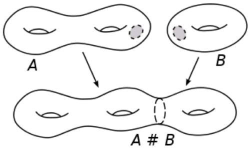




What do punctures do to Euler characteristic?



What do connected sums do to Euler characteristic?



What Euler characteristics are possible?

For surfaces without boundary?

$$S^2$$

$$\begin{array}{cc} T^2 & P^2 \\ T^2 \# T^2 & P^2 \# P^2 \\ T^2 \# T^2 \# T^2 & P^2 \# P^2 \# P^2 \\ T^2 \# T^2 \# T^2 \# T^2 & P^2 \# P^2 \# P^2 \# P^2 \\ \cdots & \cdots \end{array}$$

For surfaces with boundary?

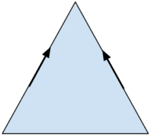
Classify surfaces by Euler characteristic (and orientability)

What surfaces have

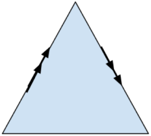
- Euler characteristic of 2?
- Euler characteristic of 1?
- Euler characteristic of 0?
- Euler characteristic of -1?

What surface is this?

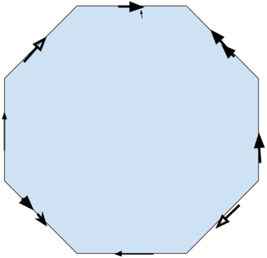
What topological surfaces do each of these figures represent?

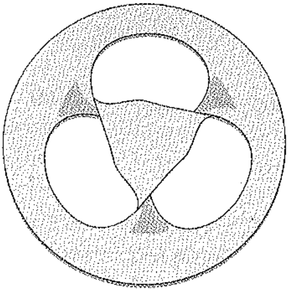


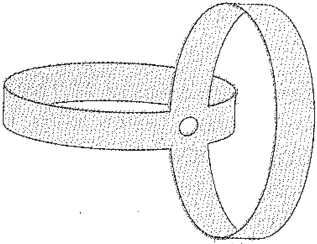
A



B







Homework

1. What topological surface are shown in the figures on the previous page? Hint: find the Euler number and count the number of boundary curves (if any). You might also need to decide if the surface has any orientation reversing paths or not. For each surface, write your answer by giving the familiar name of the surfaces if possible or by describing it as a connected sum of tori or projective planes with a certain number of punctures.