Part XXII

Orbifolds

The goal for this lesson is to fold up wallpaper patterns into orbifolds and study their curvature.

Folding up wallpaper patterns

Imagine folding a wallpaper pattern along mirror lines and rolling it up at rotation points so that all points on the paper with the same pattern are glued together, or identified as a single point. This resulting glued up object is called an orbifold. For example, the orbifold corresponding to this pattern looks like a cone with a corner.

You fold.

Pick a wallpaper pattern and find its orbifold.

- Roll your pattern around gyration points to create cone points.
- Fold your pattern along mirror lines to create boundary edges. (If folding makes the paper too thick, you can just cut along mirror lines instead.)
- Sometimes it is easiest to start with a fundamental domain and glue (or tape) together the edges that have the same part of the pattern on them.

Calculate the total curvature of your orbifold.

Curvature at gyroscope and kaleidoscope points

If a wallpaper pattern has a gyration point of order n, what will the curvature be at the resulting cone point?

If a wallpaper pattern has a kaleidoscope point with n mirrors, what will the curvature be at the resulting boundary corner?

Total curvature of wallpaper orbifolds

- Is it possible to have an orbifold with total curvature -720 degrees? Total curvature -360 degrees?

- What numbers could be possible for the total curvature of an orbifold made from a wallpaper pattern?

Topology of wallpaper orbifolds

- What numbers are possible for the Euler number of an orbifold made from a wallpaper pattern?

- What topological surfaces are possible for the underlying topology of an orbifold made from a wallpaper pattern?

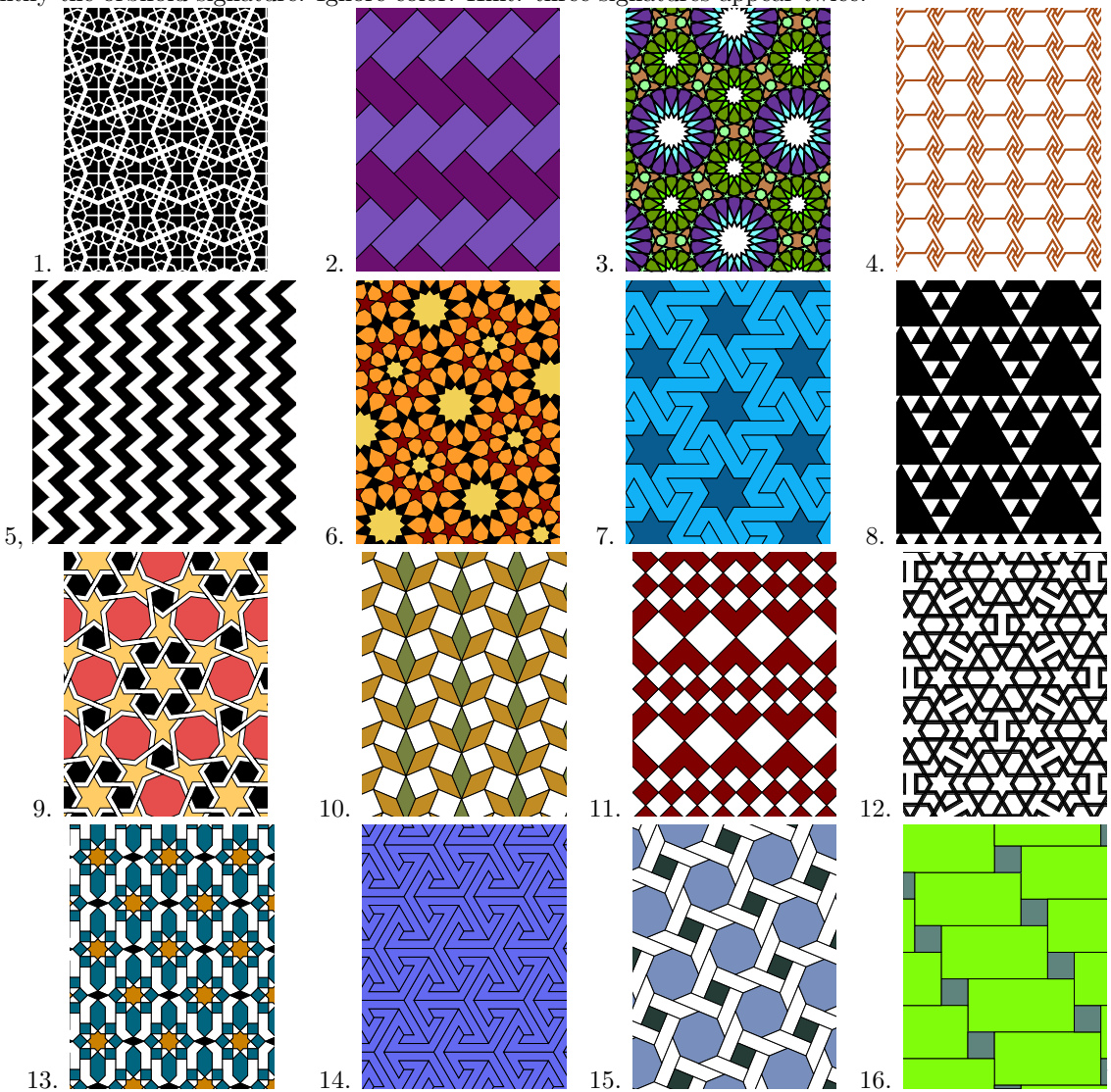
Non-orientable orbifolds

Consider the orbifolds that we have made that are non-orientable. What do their wallpaper patterns have in common?

What is the relationship between reflections, glide reflections, and non-orientable orbifolds?

More examples for building orbifolds

Identify the orbifold signature. Ignore color. Hint: three signatures appear twice.



Homework

1. For two of the patterns on the previous page of patterns, do the following:
   - Describe its orbifold (e.g. a sphere with cone points or a disk with corners, or a disk with corners and a cone point, or a Mobius band, etc.
   - Take a picture of the orbifold or draw a careful sketch, or draw its gluing diagram (handy for those that are difficult to build because they are projective planes or the like).
   - Calculate its curvature based on its Euler number.
   - Calculate its curvature from "first principles" by finding the angle defect at each cone point and corner. Show work. (You should get the same answer as the previous answer.)