Part IV

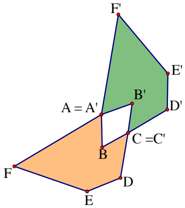
Four (or Five) Types of Isometries

Goal: Prove that there are only four (or five) isometries of the plane:

1.
2.
3.
4.

Polygons that share three vertices

A *vertex* of a polygon is a corner where two edges meet.



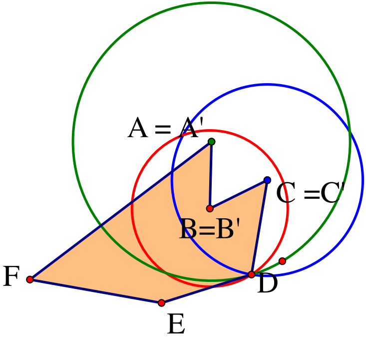
These two polygons share two vertices.

- Is it possible to draw two congruent polygons in the plane that share three corresponding vertices but don't share all their vertices?
- What about for polyhedra in 3-dimensional space?

Polygons and circles

Theorem

If two congruent polygons share three corresponding vertices that are not colinear, then they share all vertices.



Two congruent polygons that share three corresponding vertices (that are not colinear) share all vertices.

Proof.

- Suppose we have two congruent polygons that share three corresponding vertices A, B, and C.
- If there is a fourth vertex D on the first vertex, figure out how far it is from each of A, B, and C. Say it is a units from A, b units from B, and c units from C.
- Vertex D must lie on the intersection of the three circles centered at A, B, and C of radii a, b, and c, respectively.
- As long as A, B, and C are not colinear, then there is only one point of intersection of the three circles.
- Since vertex D' on the second polygon is also distance a from A, distance b from B, and distance c from C, it must also be on the intersection of the three circles.
- So vertex D' on the second polygon must coincide with vertex D on the first polygon.

There are no other isometries out there

There are no other isometries of the plane besides:

- translations
- reflections
- rotations
- glide reflections
- the identity

How do we know?

What about in 3-d?

Translations, rotations, reflections, and glides are the only isometries of the plane.

Proof.

- Three vertices of a triangle ABC can be brought onto three vertices of an isometric triangle $A'B'C'$ that using a product of one, two, or three reflections. How?

Translations, rotations, reflections, and glides are the only isometries of the plane.

Proof.

- Three vertices of a triangle ABC can be brought onto three vertices of an isometric triangle $A'B'C'$ that using a product of one, two, or three reflections. How?
- If an isometry agrees with a product of reflections on three (non-collinear) points A, B, and C, then it agrees with the product of reflections on all points. Why?

Translations, rotations, reflections, and glides are the only isometries of the plane.

Proof.

- Three vertices of a triangle ABC can be brought onto three vertices of an isometric triangle $A'B'C'$ that using a product of one, two, or three reflections. How?
- If an isometry agrees with a product of reflections on three (non-collinear) points A, B, and C, then it agrees with the product of reflections on all points. Why?
- Therefore, any isometry is a product of one, two, or three reflections.

Translations, rotations, reflections, and glides are the only isometries of the plane.

Proof.

- Three vertices of a triangle ABC can be brought onto three vertices of an isometric triangle $A'B'C'$ that using a product of one, two, or three reflections. How?
- If an isometry agrees with a product of reflections on three (non-collinear) points A, B, and C, then it agrees with the product of reflections on all points. Why?
- Therefore, any isometry is a product of one, two, or three reflections.
- If only one reflection is needed, then the isometry is a reflection.

Translations, rotations, reflections, and glides are the only isometries of the plane.

Proof.

- Three vertices of a triangle ABC can be brought onto three vertices of an isometric triangle $A'B'C'$ that using a product of one, two, or three reflections. How?
- If an isometry agrees with a product of reflections on three (non-collinear) points A, B, and C, then it agrees with the product of reflections on all points. Why?
- Therefore, any isometry is a product of one, two, or three reflections.
- If only one reflection is needed, then the isometry is a reflection.
- If exactly two reflections are needed, then the isometry is:
- If exactly three reflections are needed, then the isometry is:
- Why is the last statement true?

Why is a product of three reflections always a reflection or a glide reflection?

Proof that product of three reflections always a reflection or a glide reflection, part 1

- Suppose we have an isometry that is a product of three reflections through mirrors $m1$, $m2$, and $m3$.

Proof that product of three reflections always a reflection or a glide reflection, part 1

- Suppose we have an isometry that is a product of three reflections through mirrors $m1$, $m2$, and $m3$.
- If $m1$, $m2$, and $m3$ are all parallel, then the product of the reflections is a ____________________ . Why?

Proof that product of three reflections always a reflection or a glide reflection, part 1

- Suppose we have an isometry that is a product of three reflections through mirrors $m1$, $m2$, and $m3$.
- If $m1$, $m2$, and $m3$ are all parallel, then the product of the reflections is a ____________________ . Why?
- If $m2$ and $m3$ intersect, then reflection through $m2$ and then $m3$ is a rotation with rotocenter at the intersection of $m2$ and $m3$.
- So we can think of our isometry as reflection through $m1$ followed by rotation around this intersection point.

Proof that product of three reflections always a reflection or a glide reflection, part 1

- Suppose we have an isometry that is a product of three reflections through mirrors $m1$, $m2$, and $m3$.
- If $m1$, $m2$, and $m3$ are all parallel, then the product of the reflections is a ______________________ . Why?
- If $m2$ and $m3$ intersect, then reflection through $m2$ and then $m3$ is a rotation with rotocenter at the intersection of $m2$ and $m3$.
- So we can think of our isometry as reflection through $m1$ followed by rotation around this intersection point.
- But if we rotate $m2$ and $m3$ around this intersection point, we'll still get the same rotation with the same rotocenter.
- So rotate $m2$ and $m3$ around their intersection point until $m2$ is perpendicular to $m1$.

Proof that product of three reflections always a reflection or a glide reflection, part 2

- Now our isometry is reflection through $m1$, then $m2$, then $m3$, and $m1$ is perp to $m2$.

Proof that product of three reflections always a reflection or a glide reflection, part 2

- Now our isometry is reflection through $m1$, then $m2$, then $m3$, and $m1$ is perp to $m2$.
- So reflection through $m1$ then $m2$ is the same as rotation by ____________ degees with rotocenter at the intersection of $m1$ and $m2$.

Proof that product of three reflections always a reflection or a glide reflection, part 2

- Now our isometry is reflection through $m1$, then $m2$, then $m3$, and $m1$ is perp to $m2$.
- So reflection through $m1$ then $m2$ is the same as rotation by ____________ degees with rotocenter at the intersection of $m1$ and $m2$.
- If we rotate $m1$ and $m2$ around their intersection point, we still get the same rotation.
- So rotate $m1$ and $m2$ around their intersection point until $m2$ is parallel to $m3$.

Proof that product of three reflections always a reflection or a glide reflection, part 2

- Now our isometry is reflection through $m1$, then $m2$, then $m3$, and $m1$ is perp to $m2$.
- So reflection through $m1$ then $m2$ is the same as rotation by ____________ degees with rotocenter at the intersection of $m1$ and $m2$.
- If we rotate $m1$ and $m2$ around their intersection point, we still get the same rotation.
- So rotate $m1$ and $m2$ around their intersection point until $m2$ is parallel to $m3$.
- Now we have $m2$ and $m3$ parallel, and $m1$ perpendicular to both.

Proof that product of three reflections always a reflection or a glide reflection, part 2

- Now our isometry is reflection through $m1$, then $m2$, then $m3$, and $m1$ is perp to $m2$.
- So reflection through $m1$ then $m2$ is the same as rotation by ____________ degees with rotocenter at the intersection of $m1$ and $m2$.
- If we rotate $m1$ and $m2$ around their intersection point, we still get the same rotation.
- So rotate $m1$ and $m2$ around their intersection point until $m2$ is parallel to $m3$.
- Now we have $m2$ and $m3$ parallel, and $m1$ perpendicular to both.
- This means we reflect through $m1$ and then translate in direction of $m1$.
- This is exactly a glide reflection!!

Proof that product of three reflections always a reflection or a glide reflection, technical details

There are a few small details to worry about

- If $m1$ and $m2$ and $m3$ all intersect in the same point, then we need to modify the argument:
  - Rotate $m2$ and $m3$ around their intersection point until $m2$ is on top of $m1$.
  - Then we have reflection through the same mirror twice, followed by rotation through $m3$.
  - Reflecting through the same mirror twice does nothing.
  - So this is just a reflection through $m3$.
- If $m2$ and $m3$ are parallel, we need to modify the argument:
  - Start by rotating $m1$ and $m2$ around their intersection point until $m2$ intersects $m3$ and continue as before.

Homework

1. Give an algorithm (step by step instructions) for how to write ANY isometry as a product of at most three reflections.