Angles, Arclength, Area of Sectors, and Radial Speed

After completing this section, students should be able to:

- Convert between angles measured in degrees and radians.
- Convert between angles measured in degrees-minutes-seconds and angles measured in degrees as a decimal.
- Given a sector of a circle, and two of the following three properties – the circle's radius, the angle of the sector, and the arclength of the sector – compute the third quantity.
- Given a sector of a circle, and two of the following three properties – the circle's radius, the angle of the sector, and the area of the sector – compute the third quantity.
- Given two of the following three things – angular speed, linear speed for a point on the outside of the circle, circle radius – compute the third .
- Convert between angular speed and number of rotations per time unit.

Angles are commonly measured using either or

For degrees

- a full circle measures
- a half circle measures

For radians:

- a full circle measures
- a half circle measures

Note. To convert between degrees and radians, it is handy to use the fact that ____________ degrees equals ____________ radians.

Example. Convert $-135°$ to radians.

Example. Convert $\frac{5\pi}{4}$ radians to degrees.

Example. Convert 7 radians to degrees.

Sometimes, angles are given in terms of degrees, minutes, and seconds, as in $32°17'25''$

A minute is ...

A second is ...

Example. Convert $32°17'25''$ to a decimal number of degrees.

Example. Convert $247.3486°$ to degrees, minutes, and seconds.

Arclength and Area of Sectors

The circumference of a circle is given by the formula ...

Example. A circular pool has a radius of 8 meters. Find the arclength spanned by a central angle of 2.5 radians.

The arclength is related to the angle it spans according to the formula ...

The area of a circle is given by the formula ...

Example. Find the area of a sector of a circle of radius 10 meters, that spans an angle of $\frac{\pi}{6}$ radians.

The area of a sector is related to the angle is spans according to the formula ...

Linear and Radial Speed

Consider a spinning wheel.

Definition. The *angular speed* is ...

and has units of ...

Definition. The *linear speed* is ...

and has units of ...

Example. A ferris wheel with radius 20 m is making 1 revolution every 2 minutes. What is its angular speed? The linear speed of a point on its rim?

The linear speed is related to the angular speed by the formula ..

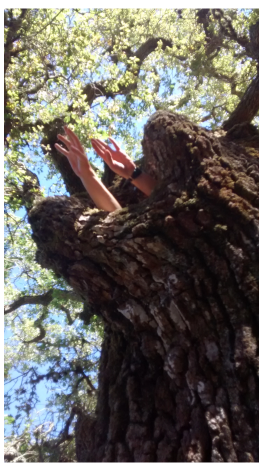
END OF VIDEOS

Example. Convert between degrees and radians:

degrees	radians
3°	
400°	
	$-\frac{\pi}{8}$
	1

Example. A geocache is hidden at coordinates $N\ 35°55'7''N$ and $W\ 79°02'59''$. Convert the coordinates to decimal degrees.

Example. This geocache is located at $N38.00958°$ and $W122.56847°$. Convert the coordinates to degrees, minutes, and seconds.



NEXT TIME, MAYBE TRY SOMETHING LIKE THIS:

Example. Which formulas give the relationship between angle and arclength and why do they hold? (Multiple choice)

Example. Big Ben is the largest clock tower in England and overlooks the Houses of Parliament and the Thames River. The minute hand of the clock measures 36 feet in length. How far does the tip of the minute hand move in 20 minutes?

NEXT TIME SWITCH THIS so that you start with the rectangular pizza, and the kid can eat 3 slices. What angle wedge can the kid eat from a 12 inch round pizza? and skip the later problems. Note that a 12 inch pizza has a diameter of 6 inches.

Example. Suppose my kid can eat two slices of pizza from a 12 inch pizza cut into 6 slices. I make homemade pizza on a rectangular cookie sheet that is 10 inches by 15 inches and I cut it into 8 rectangular pieces. How many slices of this homemade pizza should I pack in my kid's lunch?

Suppose a kid can eat 3 slices of pizza from a 10″ x 15″ rectangular pizza. What angle wedge of pizza can they eat from a 16 inch pizza?

Extra Example. A sector has area 4 m^2 and arclength $1.5m$. Is it possible to find the radius of the circle it comes from? The angle it subtends?

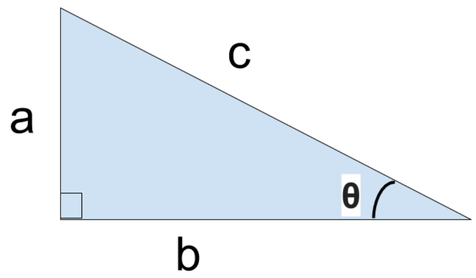
Example. Which is greater: the speed at which a point on the equator of the earth is moving due to the earth's rotation or the speed of sound? Ignore the motion of the earth around the sun.

Right Angle Trigonometry

After completing this section, students should be able to

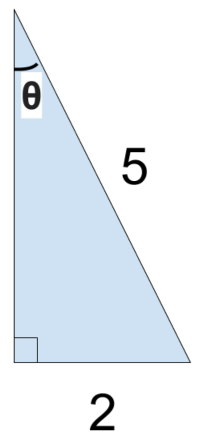
- Compute sin(θ), cos(θ), tan(θ),sec(θ), csc(θ), and cot(θ) given at least two sides of a right triangle with angle θ.
- Explain why the computation of cos(θ) is the same for any right triangle with angle θ, no matter how big or small it is (and likewise the particular right triangle used doesn't matter for the other trig functions, only lthe angle θ).
- Use geometry to compute sin, cos and tan of the special angles 30°, 45° and 60°.

For a right triangle with sides a, b, and c and angle θ as drawn,



we define

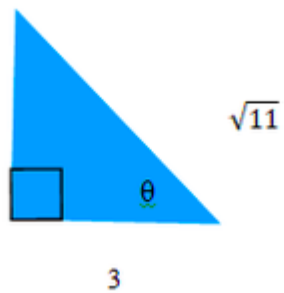
Example. Find the exact values of all six trig functions of angle θ in this right triangle.



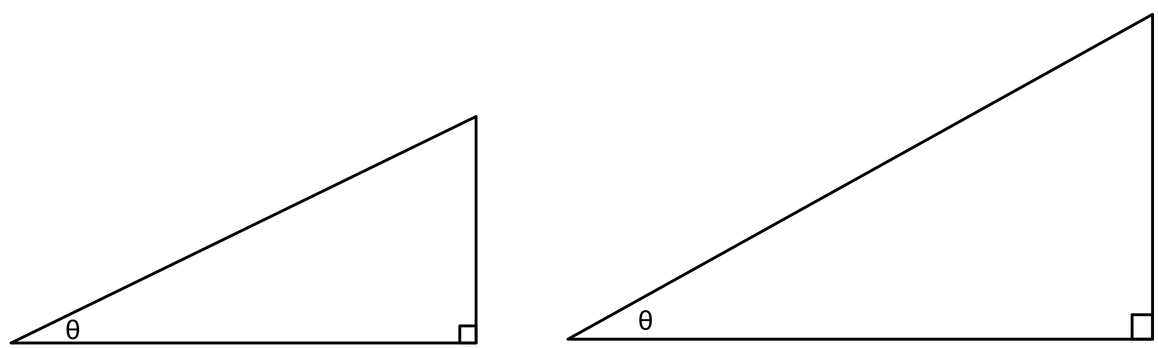
Example. A kite is flying at an angle of elevation of 75°, with 100 m of kite string let out. How high is the kite?

END OF VIDEO

Example. Find the exact values of all six trig functions of angle θ:



Suppose you have two different right triangles with the same angle θ. Suppose Liliana computes $\sin(\theta)$ using the triangle on the left and Rachel computes it using the triangle on the right. Will they get the same answer? Explain.



Example. A 10-foot ladder makes a 70^o angle with the ground while leaning against a building. How far is the bottom of the ladder from the base of the building?

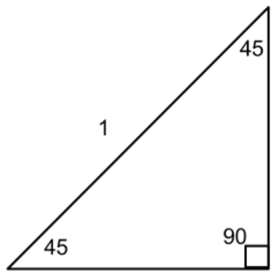
NOTE for next time: this is way too similar to the video problem. Do one where students have to figure out angles from other angles or have to solve for a variable in the denominator or have to deal with an angle of elevation or have to use something besides tan.

Unit Circle

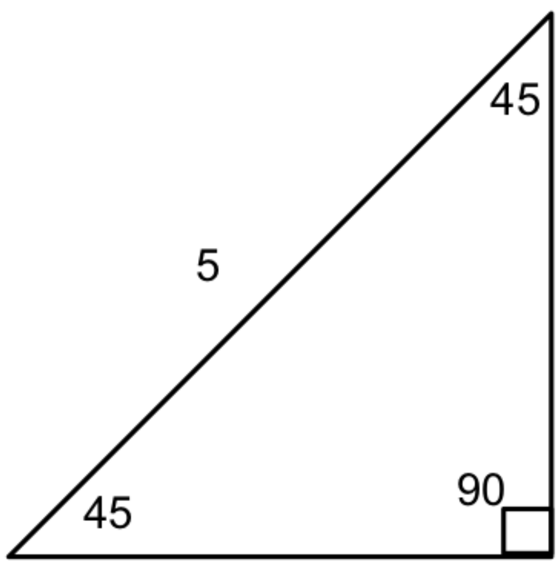
After completing this section, students should be able to:

- Compute sin, cos and tan of the angles $45°$, $30°$, and $60°$ using right triangles and geometry.
- Use a calculator to evaluate sin, cos, and tan of other angles.
- Describe the relationship between sin and cos and the coordinates of a point on a unit circle.
- Explain why the right triangle definition of cos and sin (adj/hyp and opp/hyp) corresponds to the unit circle definition (x-coord and y-coord of terminal point) for angles between $0°$ and $90°$.
- Fill in a blank unit circle.
- Use the unit circle to evaluate trig functions on special angles.

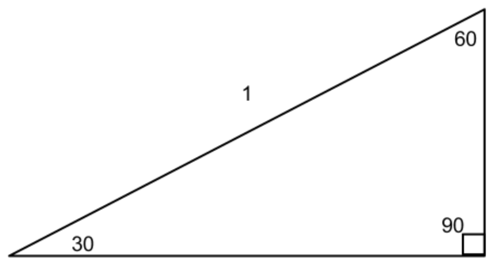
Example. Without using a calculator, find sin(45°) and cos(45°) using a right triangle with hypotenuse 1.



Example. Without using a calculator, find sin(45°) and cos(45°), and tan(45°) using a right triangle with hypotenuse 5.



Example. Without using a calculator, find $\sin(30°)$ and $\cos(30°)$.



Example. Without using a calculator, find $\sin(60°)$ and $\cos(60°)$.

To summarize:

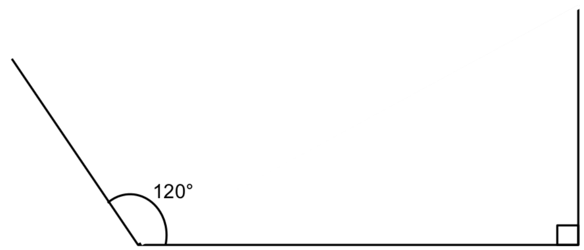
angle θ in degrees	angle θ in radians	$\cos(\theta)$	$\sin(\theta)$
$30°$			
$45°$			
$60°$			

Note. For other angles, it is more difficult to compute an exact answer for $\sin(\theta)$ and $\cos(\theta)$, and we will usually use a calculator to get a decimal approximation.

Example. Use a calculator to compute $\sin(40°)$ and $\cos(\frac{\pi}{5})$

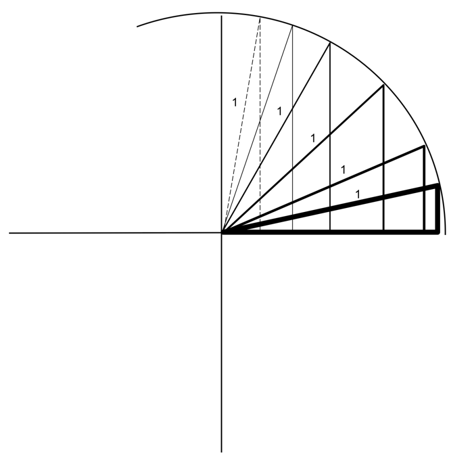
Note. Up to now we have defined sine, cosine, and tangent in terms of right triangles. For example, to find sin(14°) ...

What goes wrong if we try to compute sin(120°) using a right triangle?



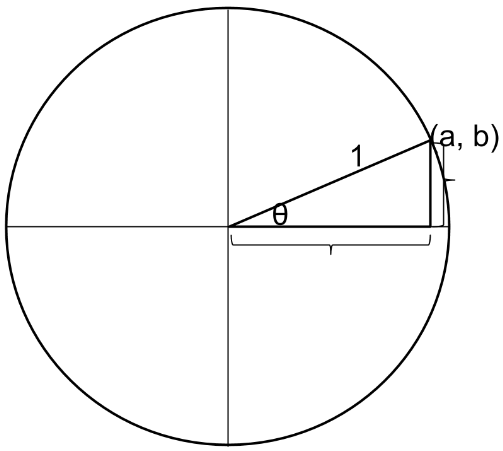
Instead of using a right triangle to define sin(120°) and cos(120°), we need to use a *unit circle*.

Definition. The *unit circle* is ...



In the figure, a right triangle is drawn in the unit circle. It's top vertex has coordinates (a, b). In terms of a and b,

- How long is the base of the triangle?
- How long is its height?



Using the triangle definition of sine and cosine, what are $\sin(\theta)$ and $\cos(\theta)$, in terms of a and b?

$\cos(\theta) =$ $\sin(\theta) =$ $\tan(\theta) =$

Definition. For angles θ that can't be part of a right triangle we find the point on the unit circle at angle θ and **define**

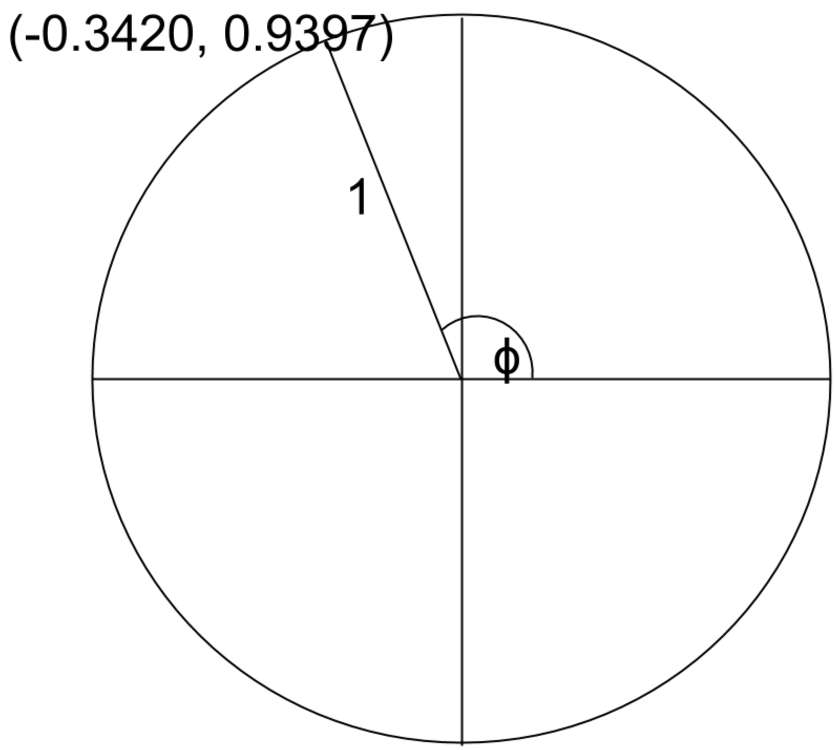
$\cos(\theta) =$ $\sin(\theta) =$ $\tan(\theta) =$

Example. For the angle ϕ drawn,

$\sin(\phi) =$________

$\cos(\phi) =$________

$\tan(\phi) =$________



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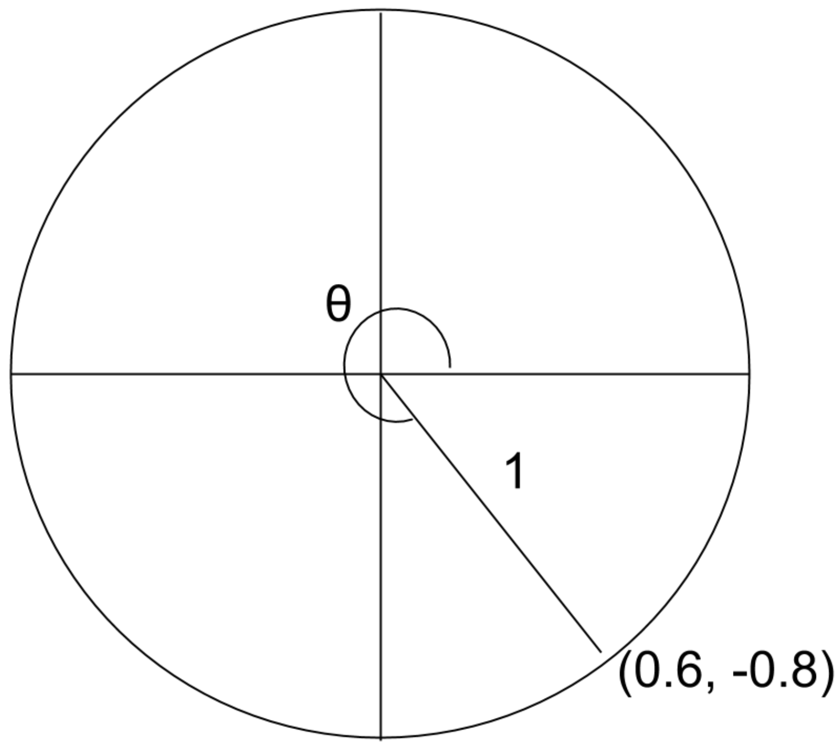
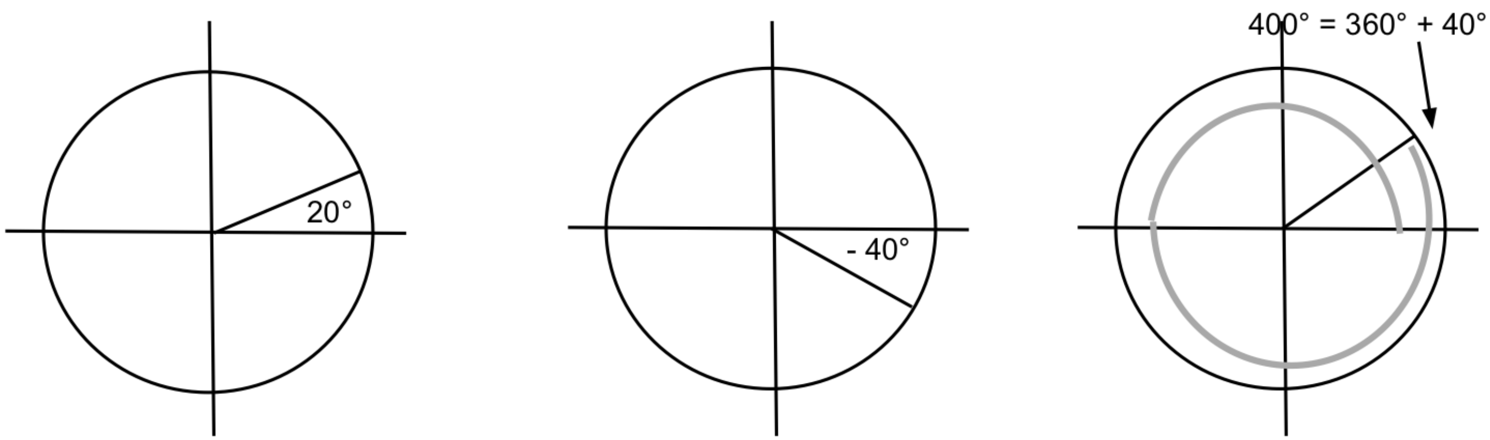
Review. Find $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ for the angle drawn on the unit circle below.



Note. When using the unit circle, there are some conventions for drawing angles.

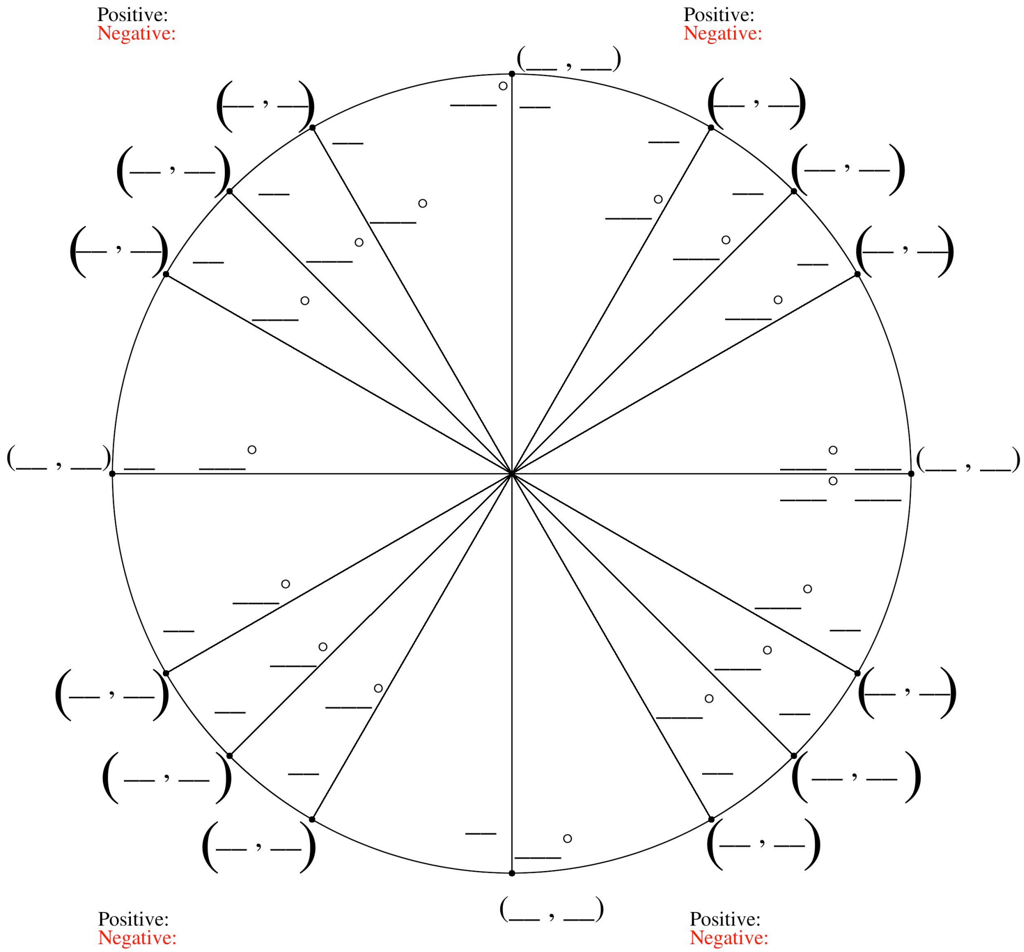
1. Measure angles in the counterclockwise direction, starting from the positive x-axis.
2. Negative angle go in the clockwise direction.
3. Angles greater than 360° wrap around the circle more than once.



Example. Use the unit circle to find sine, cosine, and tangent of the following angles, without using a calculator.

angle θ	$\cos(\theta)$	$\sin(\theta)$	$\tan(\theta)$
180°			
-90°			
5π			
225°			
$-\frac{\pi}{6}$			

Example. Without a calculator, fill out the angles and the exact x- and y-coordinates of the points on the blank unit circle.



Hint: fill out the first quadrant, and then use symmetry to fill out the other quadrants. The angles in the first quadrant are 30°, 45°, and 60°.

Example. Find the exact values:

1. $\cot(405^\circ)$
2. $\csc\left(-\frac{5\pi}{3}\right)$

Properties of Trig Functions

After completing this section, students should be able to:

- Explain why $\cos(\theta)$ is an even function and $\sin(\theta)$ is.an odd function.
- Explain why $\cos^2(\theta) + \sin^2(\theta) = 1$
- Explain why $\sin(\theta)$ and $\cos(\theta)$ are periodic with period 2π
- Calculate the values of all the trig functions at angle θ from the value of a single trig function at θ and inforamtion on which quadrant the angle is in. For example, calculate $\sin(\theta)$, $\cos(\theta)$, $\sec(\theta)$, $\csc(\theta)$ and $\cot(\theta)$ given the value of $\tan(\theta)$ and the fact that the angle is between 90° and 180°.

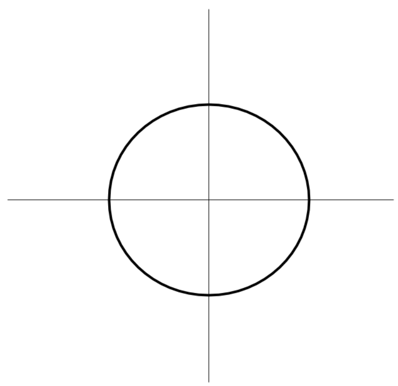
Periodic Property:

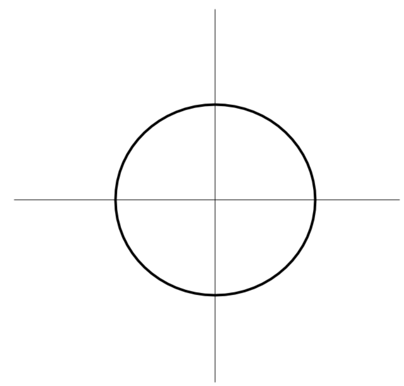
$\cos(\theta + 2\pi) =$

$\cos(\theta - 2\pi) =$

$\sin(\theta + 2\pi) =$

$\sin(\theta - 2\pi) =$





Example. Find

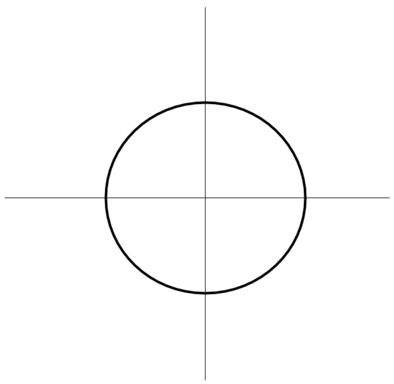
1. $\cos(5\pi)$

2. $\sin(-420°)$

Even and Odd Property:

$\cos(-\theta) =$

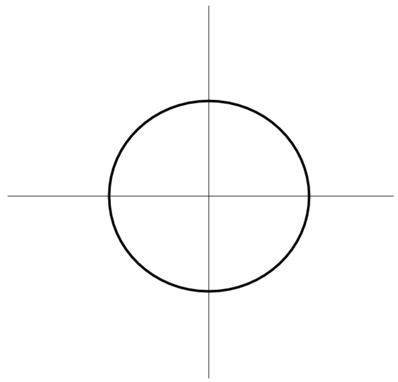
$\sin(-\theta) =$



Example. Determine if $\tan(\theta)$ is even or odd.

Pythagorean Property:

$\cos^2(\theta) + \sin^2(\theta) =$



Example. If $\sin(t) = -\frac{2}{7}$, and t is angle that lies in quadrant III, find $\cos(t)$ and $\tan(t)$.

END OF VIDEO

Question. Which of the following equations hold for all value of θ?

A. $\cos^2(\theta) + \sin^2(\theta) = 0$

B. $\cos(-\theta) = \cos(\theta)$

C. $\sin(-\theta) = \sin(\theta)$

D. $\cos(\theta + \pi) = \cos(\theta)$

Extra Example. Find the exact value of

1. $\sin(600°)$

2. $\cos(\frac{65\pi}{6})$.

3. $\tan(-135°)$

Example. If $\sin(\theta) = \dfrac{2}{5}$ and θ lies in the second quadrant, find $\tan(\theta)$.

Example. If $\tan(\theta) = \dfrac{4}{3}$, and $\cos(\theta)$ is negative, find $\cos(\theta)$ and $\sin(\theta)$.

Extra Example. Let $(\sqrt{5}, -2)$ be a point on the terminal side of angle θ. Find the exact values of $\cos(\theta)$, $\csc(\theta)$, and $\tan(\theta)$.

Example. Use your calculator to find $\cos(52°)$ and $\sin(38°)$. What relationship do you notice and why does it hold?

Property 4:

$\cos(90° - \theta) =$

$\cos(\frac{\pi}{2} - \theta) =$

$\sin(90° - \theta) =$

$\sin(\frac{\pi}{2} - \theta) =$

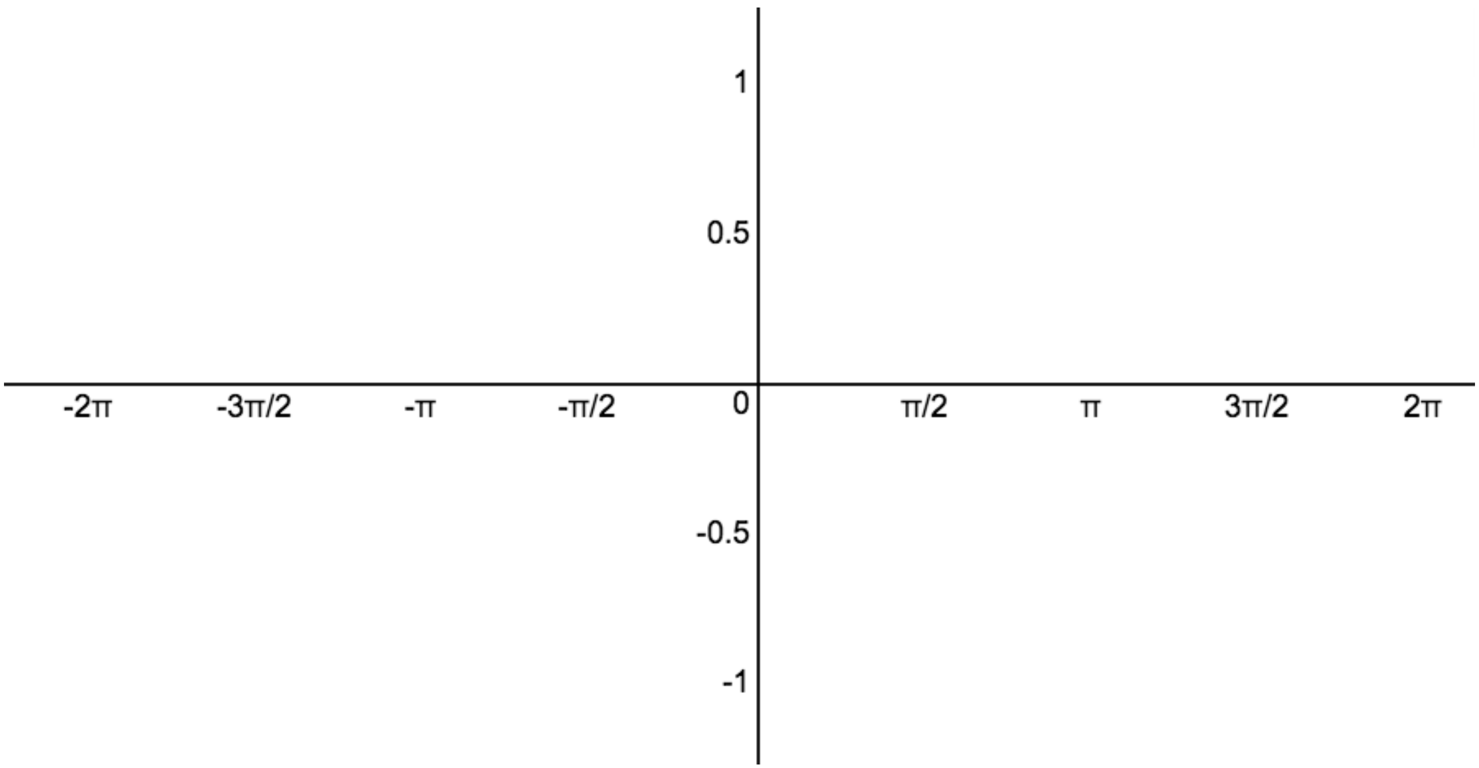
Graphs of Sine and Cosine

After completing this section, students should be able to:

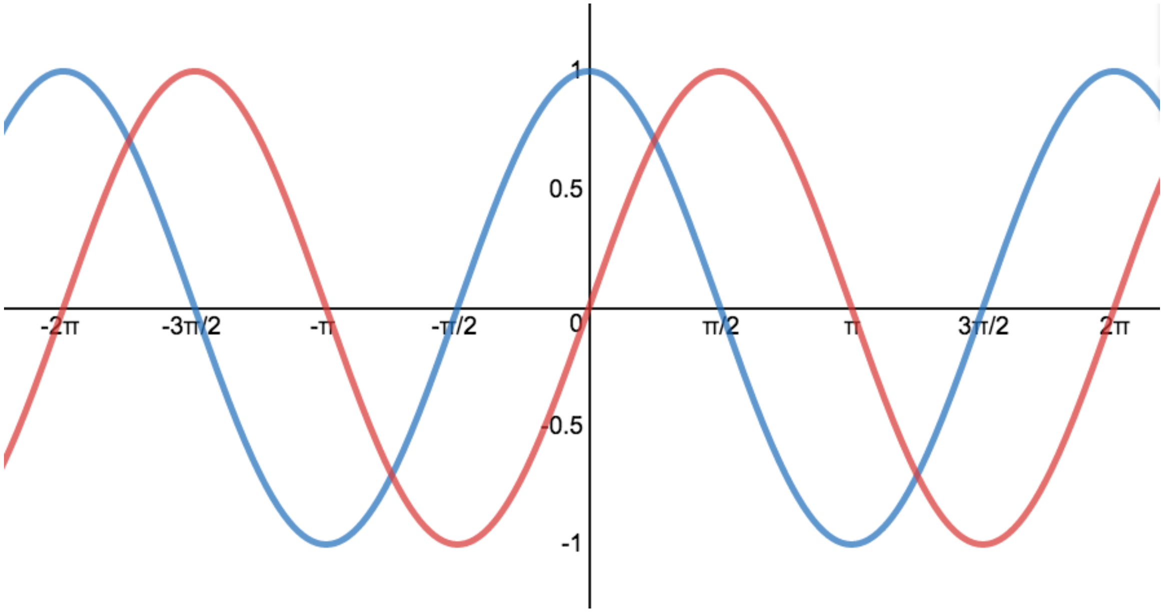
- Graph sinusoidal functions: i.e. functions of the form $y = A\sin(Bx + C) + D$ and $y = A\cos(Bx + C) + D$
- Identify the period, amplitude, midline, and horizontal and vertical shift of a sinusoidal function from its equation or from its graph.
- Write down the equation from the graph of a sinusoidal function.
- Match graphs with equations.
- Use sinusoidal functions to model real world phenomena.

Example. Graph $y = \cos(t)$ and $y = \sin(t)$, where t is in radians.

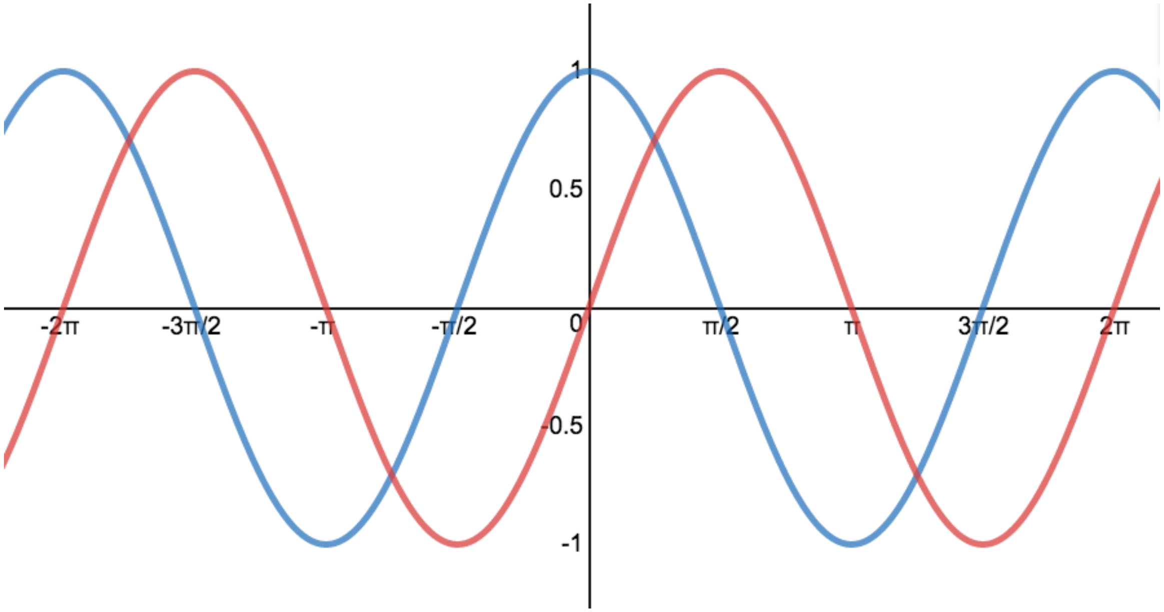
t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{12}$	2π
$\cos(t)$																	
$\sin(t)$																	



The graph of $y = \cos(x)$ is the same as the graph of $y = \sin(x)$ shifted horizontally by
$\cos(x) = \sin(x \qquad)$ $\qquad \sin(x) = \cos(x \qquad)$



- Domain
- Range
- Even / odd
- Abs max and min values

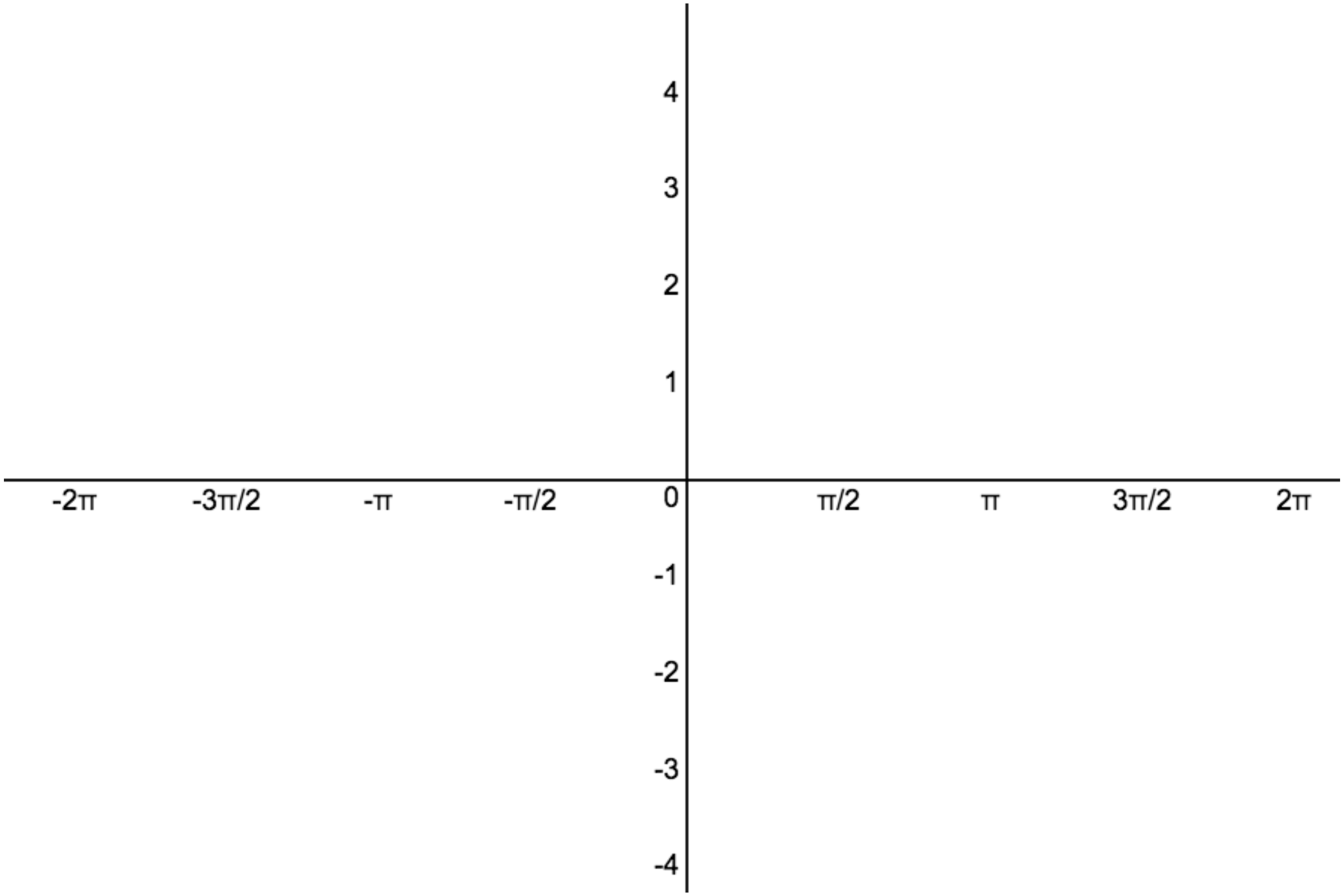


- Midline

- Amplitude

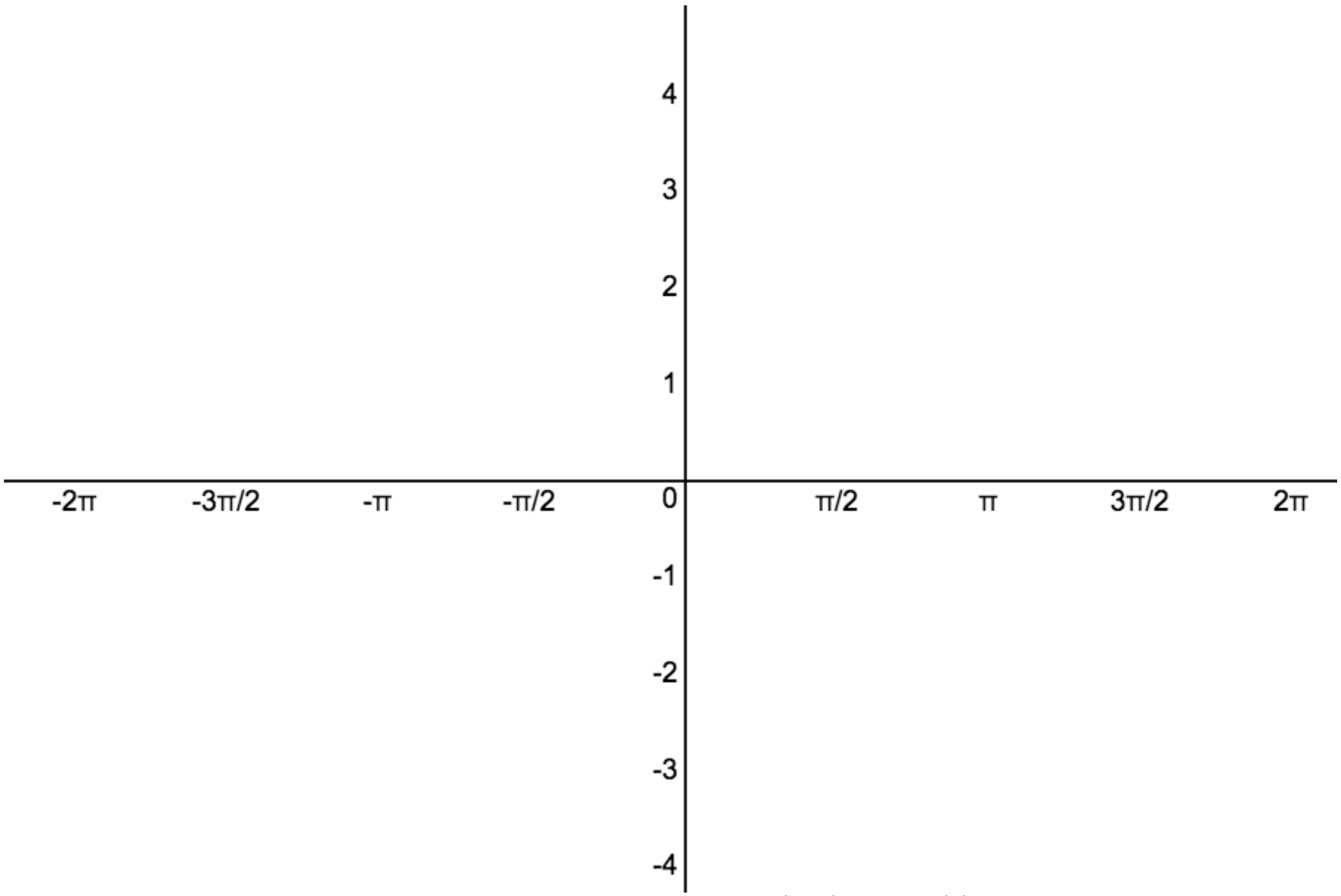
- Period

Example. Graph the function $y = 3\sin(2x)$



Example. Graph the function $y = 3\sin(2x) + 1$

Example. Graph the function $y = 3\sin\left(2\left(x - \frac{\pi}{4}\right)\right)$



Example. Graph the function $y = 3\sin\left(2\left(x - \frac{\pi}{4}\right)\right) - 1$

Summary

For the graphs of $y = A\cos(Bx - C) + D$ and $y = A\sin(Bx - C) + D$ with B positive.

- midline

- amplitude

- period

- horizontal shift

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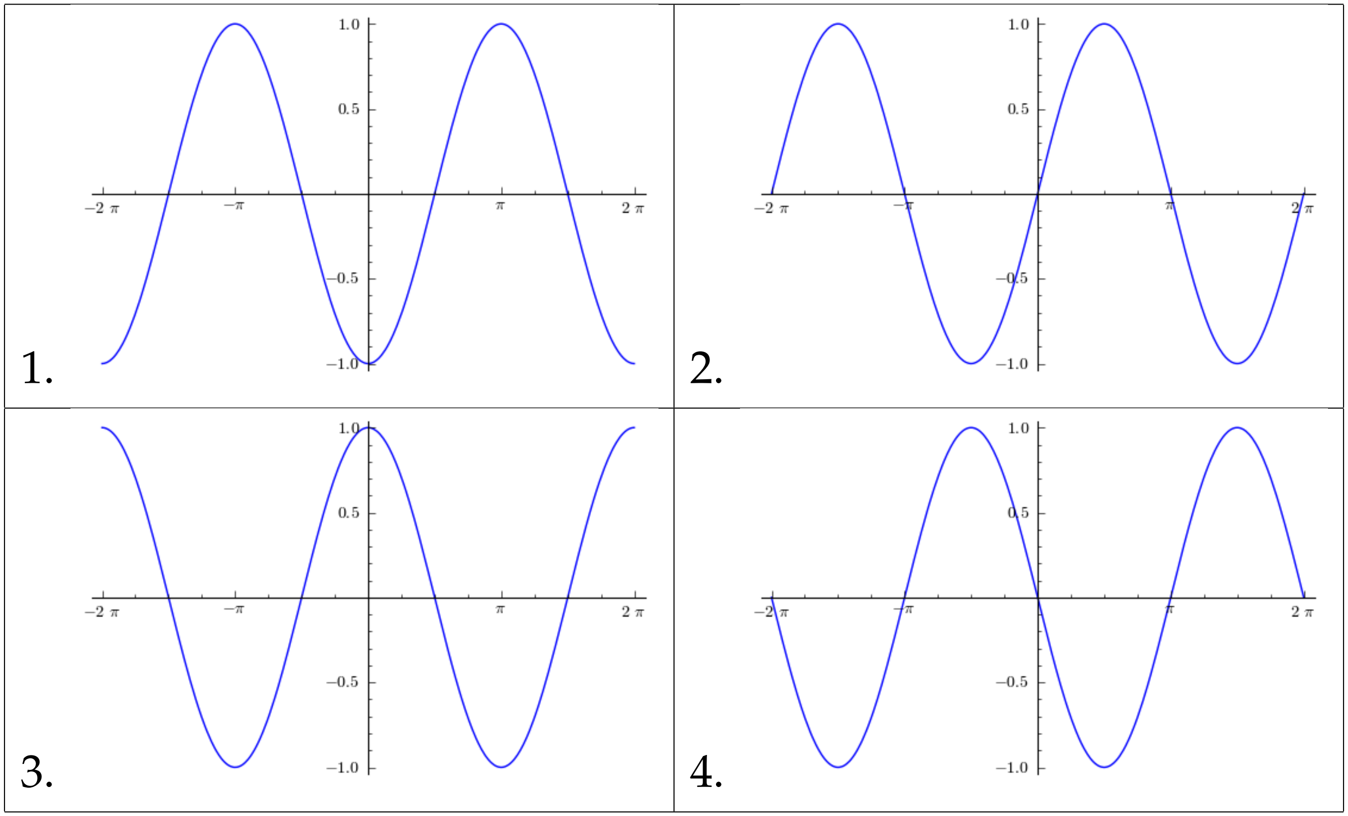
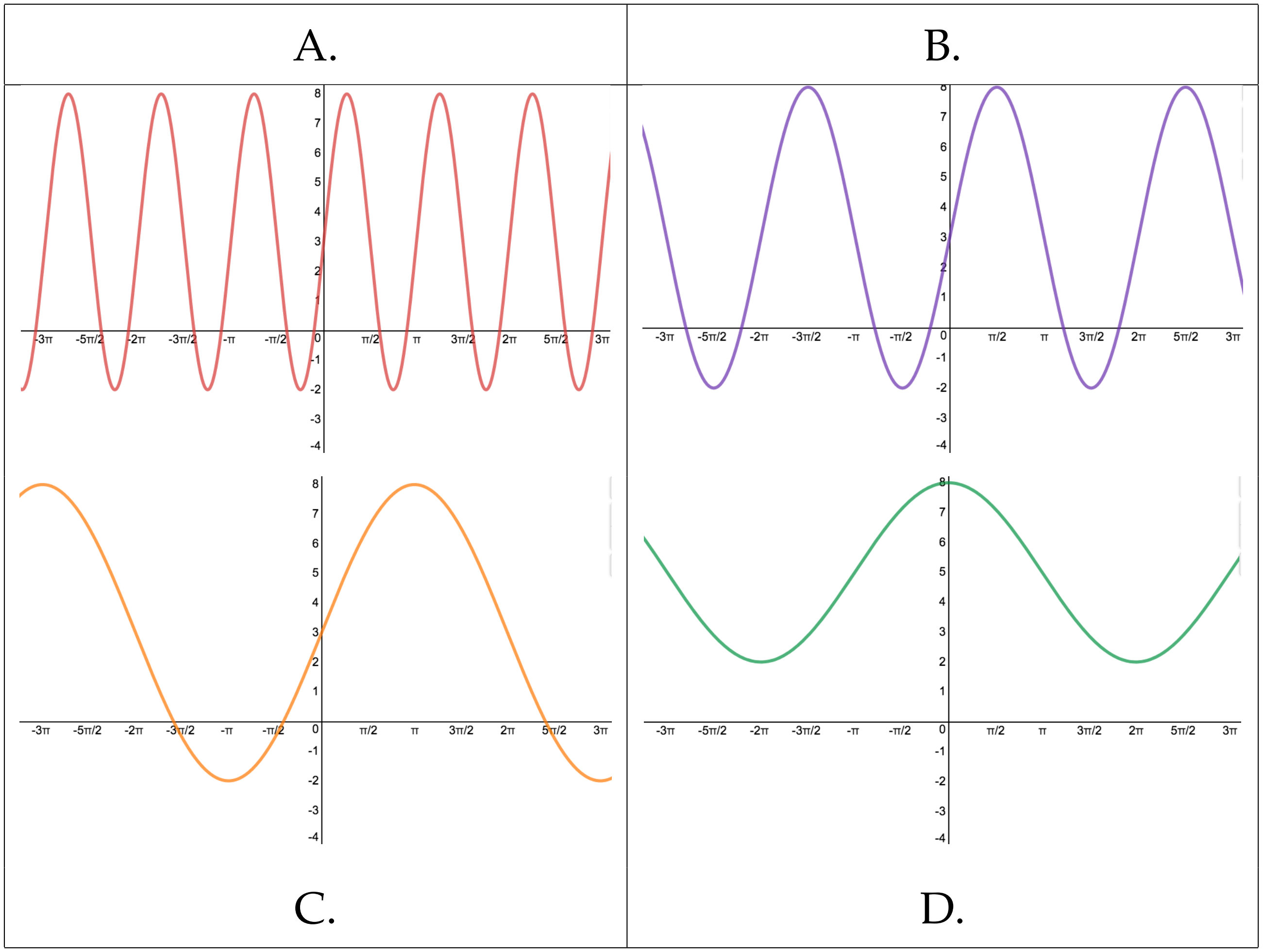
Example. Match the graphs of the sinusoidal functions to their equations.



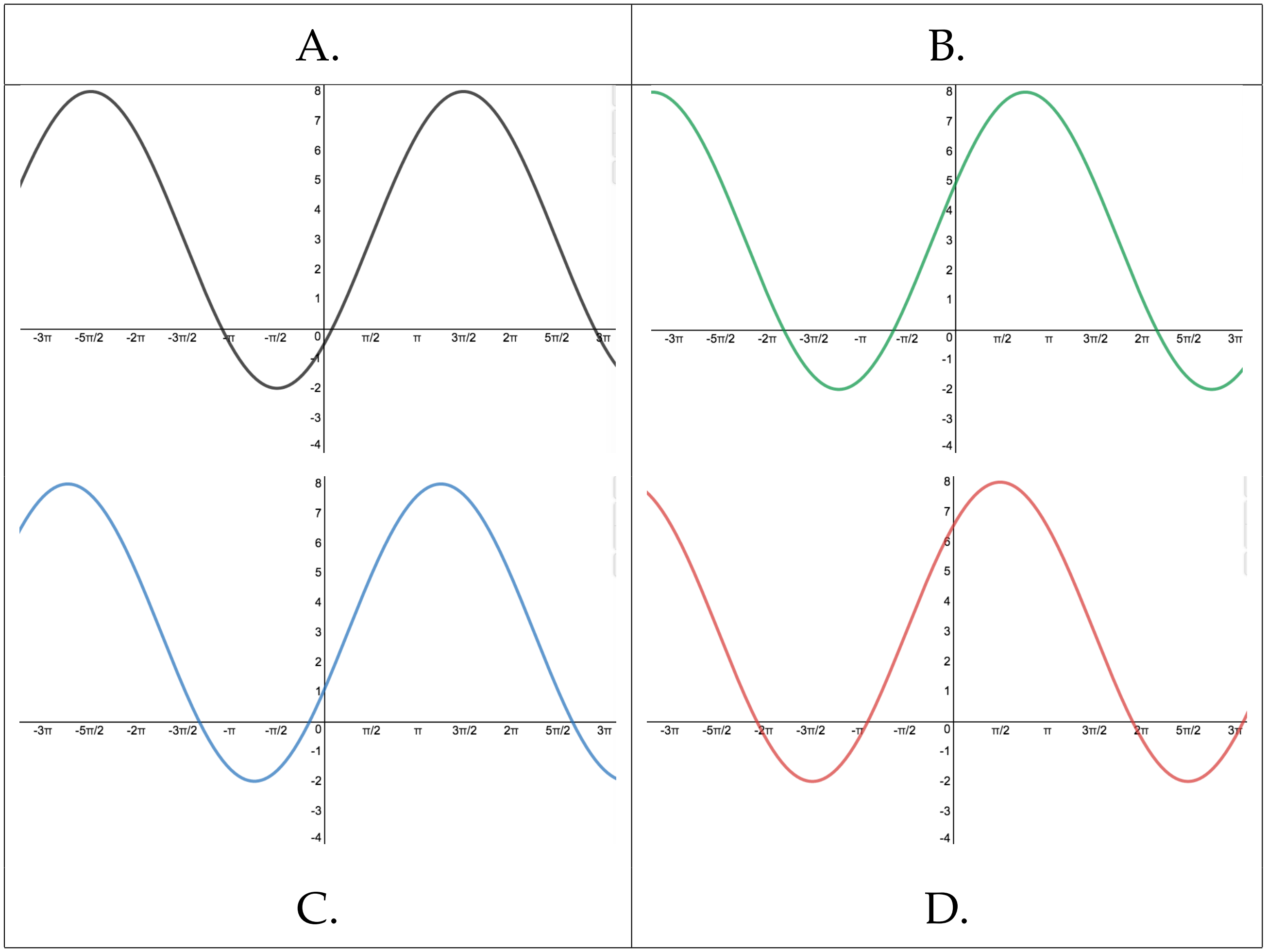
A. $y = \cos(t)$ B. $y = \sin(t)$ C. $y = -\cos(t)$ D. $y = -\sin(t)$

Example. Select the graph of $y = 5\sin\left(\frac{1}{2}x\right) + 3$



midline: amplitude: period:

Example. Select the graph of $y = 5\sin\left(\frac{1}{2}x - \frac{\pi}{4}\right) + 3$



midline: amplitude: period: phase shift:

Summary

For the graphs of $y = A\cos(Bx - C) + D$ and $y = A\sin(Bx - C) + D$ with B positive.

- midline

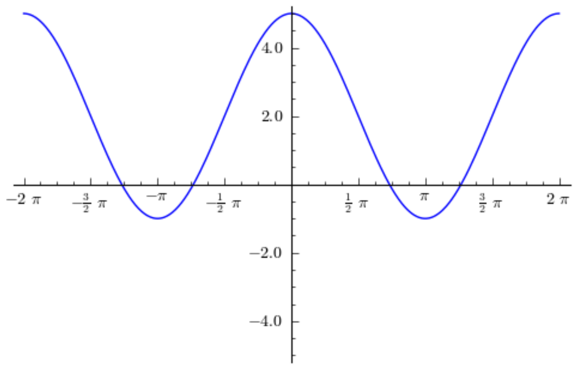
- amplitude

- period

- phase shift

Question. What if B is negative? How would you graph $y = 2\sin\left(-\frac{\pi}{2}x - \frac{\pi}{3}\right)$?

Extra Example. Find the midline, amplitude, and period for the graph:



midline:

amplitude:

period :

What is the equation for the function?

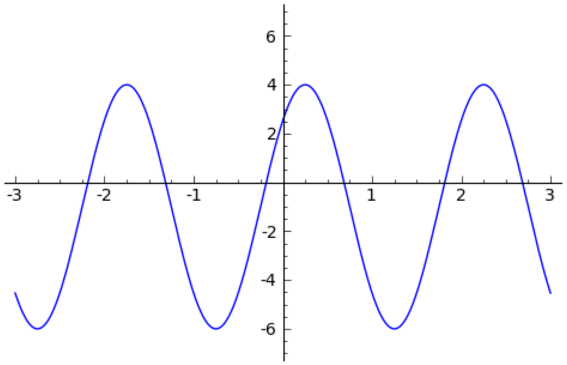
A. $y = 3\cos(x) + 2$

B. $y = 3\sin(x) + 2$

C. $y = 2\cos(x) + 3$

D. $y = \sin(3x - 2)$

Example. Find the midline, amplitude, period, and phase shift for the graph:



midline:

amplitude:

period :

phase shift:

What is the equation for the function?

A. $y = 5\sin\left(\frac{1}{2}x + \frac{1}{4}\right) - 1$

B. $y = 5\cos\left(\pi x - \frac{1}{4}\right) - 1$

C. $y = 5\sin\left(\pi x + \frac{\pi}{4}\right) - 1$

D. $y = 5\cos\left(\pi x - \frac{\pi}{4}\right) - 1$

Extra Example. In the United States, electricity cycles between 155.6 volts and -155.6 volts 60 times per second. Use cosine to model the voltage. Hint: start by drawing a rough graph of electricity over time. Then find the midline, amplitude, period, and equation.

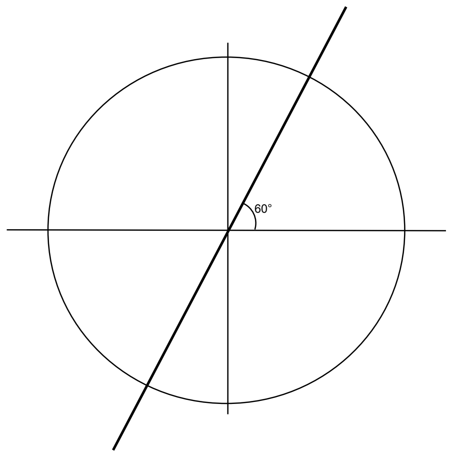
Extra Example. A Ferris wheel has a diameter of 100 feet. It turns once every 5 minutes. You start at time $t = 0$ at the bottom, which is 10 feet above the ground. Write an equation to describe your height above ground as a function of time. Hint: start by drawing a rough graph.

Graphs of Tan, Sec, Etc.

After completing this section, students should be able to:

- Find the vertical asymptotes and domain and range of transformations of tan, cot, sec, and csc.
- Draw sketches of functions of the forms $y = A\tan(Bx-C)+D$, $y = A\sec(Bx-C)+C$, etc.
- Write down the equation from the graph of transformations of tan, cot, sec, and csc.
- Match graphs of transformations of tan, cot, sec, and csc with their equations.

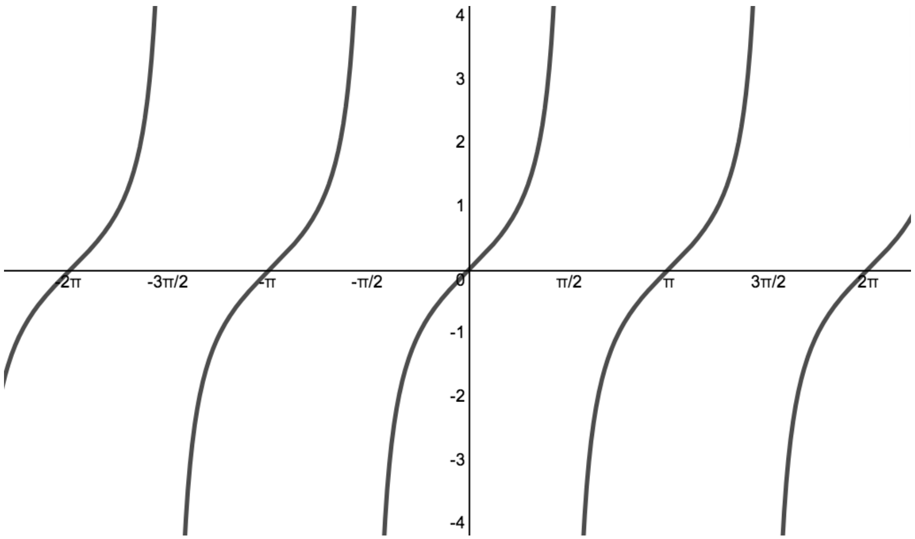
Example. What is the slope of the line at angle θ drawn below?



- The slope of the line at angle θ through the origin is ...
- When the angle is zero, the slope is ...
- As the angle increases toward $\frac{\pi}{2}$, the slope goes to ...
- As the angle goes from zero towards $-\frac{\pi}{2}$, the slope goes to ...
- At exactly $\frac{\pi}{2}$ and $-\frac{\pi}{2}$, the slope is ...

Sketch a rough graph of $y = \tan(t)$ for t between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Example. Consider this graph of $y = \tan(x)$



In the graph above, find the:

x-intercepts:

vertical asymptotes:

domain:

range:

period:

Example. Sketch a rough graph of $y = \sec(x)$ from $x = 0$ to $x = 2\pi$.

x-intercepts:

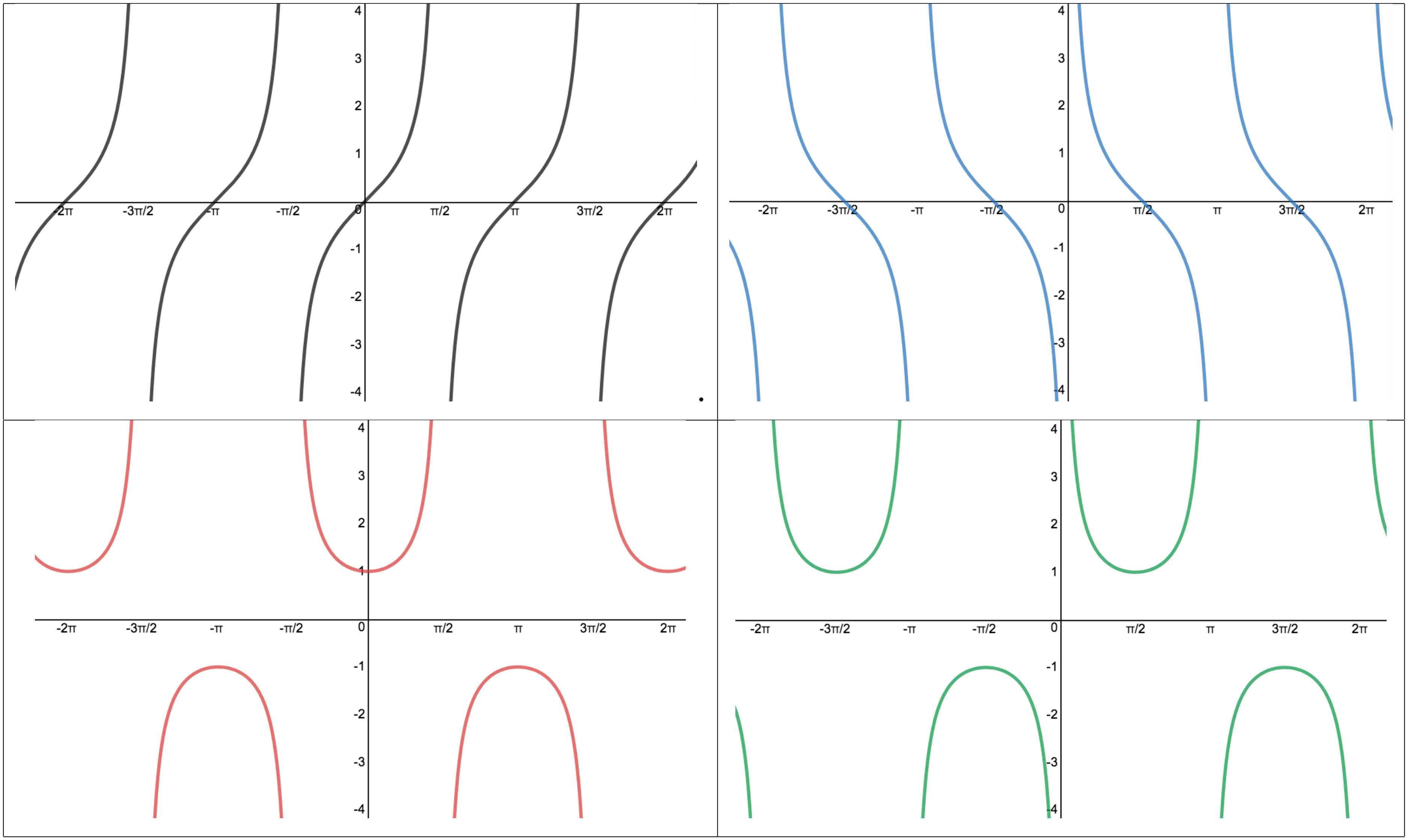
vertical asymptotes:

domain:

range:

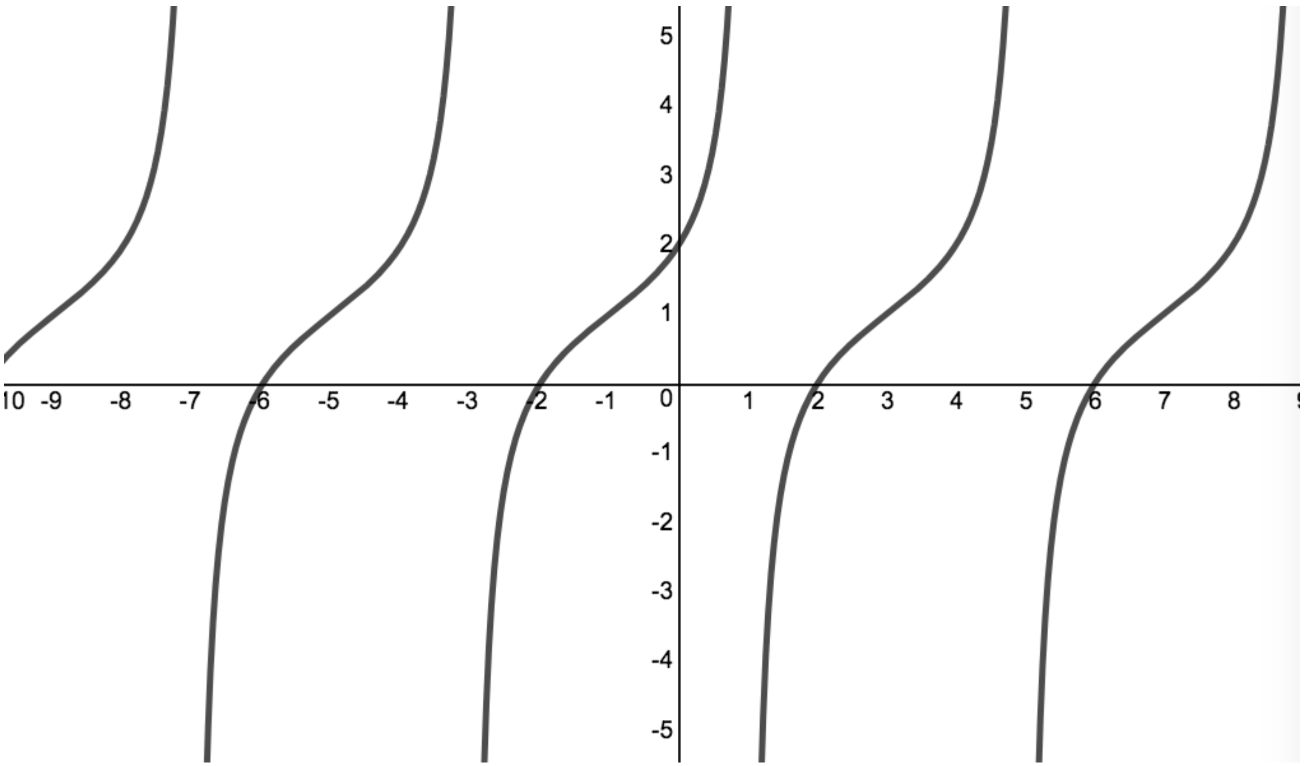
period:

Here are graphs of $y = \tan(x)$, $y = \cot(x)$, $y = \sec(x)$ and $y = \csc(x)$.



Example. Sketch a graph of the function $y = 2\csc(\pi x + \dfrac{\pi}{2}) + 1$.

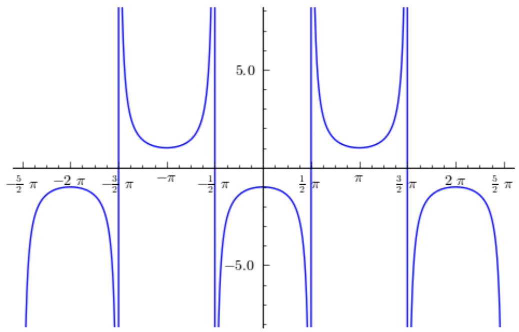
Example. Find the equation of this graph.



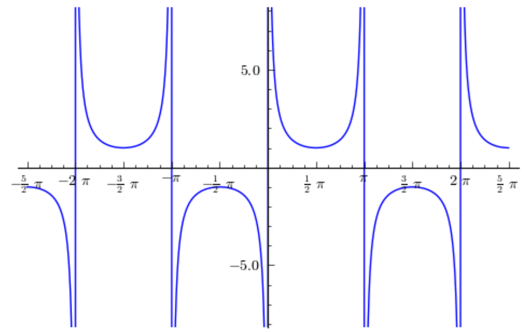
For the function $y = A\tan(Bx - C) + D$, how do the numbers A, B, C, and D affect the graph?

For the function $y = A\sec(Bx - C) + D$, how do the numbers A, B, C, and D affect the graph?

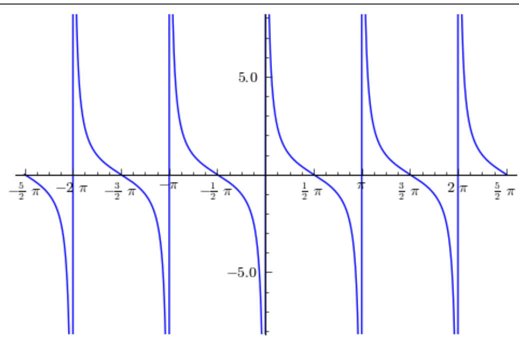
Matching



A.



B.



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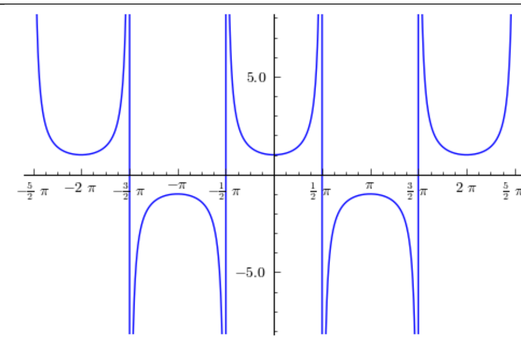

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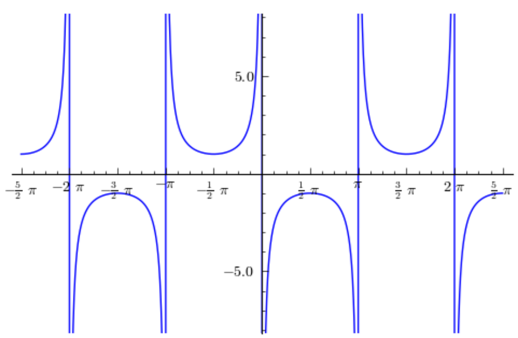
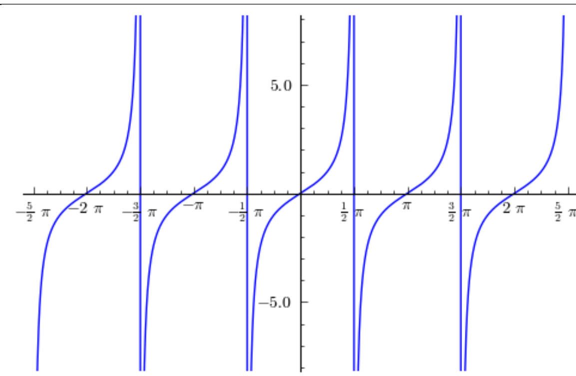


E.



F.

1. $y = \sec(x)$ 2. $y = \csc(x)$ 3. $y = -\sec(x)$ 4. $y = -\csc(x)$ 5. $y = \tan(x)$ 6. $y = \cot(x)$

Example. Where does the function $y = \cot(x)$ have vertical asymptotes? What is its domain?

Example. Where does the function $y = -\cot\left(\frac{\pi}{2}x + \frac{\pi}{8}\right) + 3$ have vertical asymptotes? What is its domain?

NEXT TIME, omit this if not needed for homework or solve it by finding one V.A. and using the period to find the others. Save the k manipulations for when we get to solving trig equations.

Example. Where does the function $y = \sec(x)$ have vertical asymptotes? What is its domain?

Example. Where does the function $y = 2\sec(3x + \frac{\pi}{6})$ have vertical asymptotes? What is its domain?

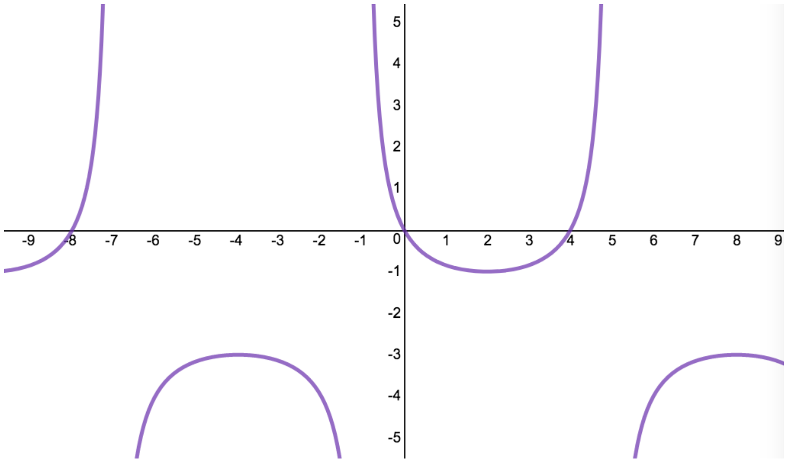
NEXT TIME, omit this if not needed for homework or solve it by finding one V.A. and using the period to find the others. Save the k manipulations for when we get to solving trig equations.

Example. Sketch the graph of $y = -\cot\left(\frac{\pi}{2}x + \frac{\pi}{8}\right) + 3$.

For the function $y = A\tan(Bx - C) + D$, how do the numbers A, B, C, and D affect the graph?

For the function $y = A\sec(Bx - C) + D$, how do the numbers A, B, C, and D affect the graph?

Example. Find the graph of this function:



A. $y = \csc(x+1) - 2$

B. $y = \sec\left(\frac{\pi}{3}x - \frac{\pi}{3}\right)$

C. $y = \csc\left(\frac{\pi}{6}x + \frac{\pi}{6}\right) - 2$

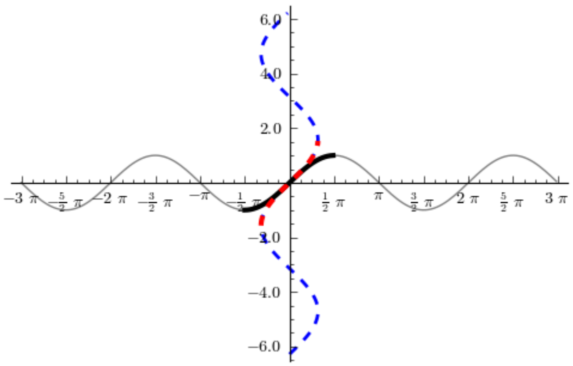
D. $y = \sec\left(\frac{\pi}{6}x - \frac{\pi}{3}\right) - 2$

InverseTrig Functions

After completing this section, students should be able to:

- State the domain and range of $f(x) = \arccos(x)$, $f(x) = \arcsin(x)$, and $f(x) = \arctan(x)$
- Find $\arccos(x)$, $\arcsin(x)$, and $\arctan(x)$ for values of x corresponding to special angles on the circle.
- Compute expressions like $\arcsin(\sin \frac{7\pi}{9})$ and $\sin(\arcsin(0.2)$ by hand.
- Rewrite trigonometric expressions like $\sec(\tan^{-1}(-\frac{4}{7}))$ and $\cos(\arctan(x))$ as numerical or algebraic expressions.

Inverse Sine Function



Restricted $\sin(x)$ **has** **Domain:** **Range:**

arcsin(x) has Domain: Range:

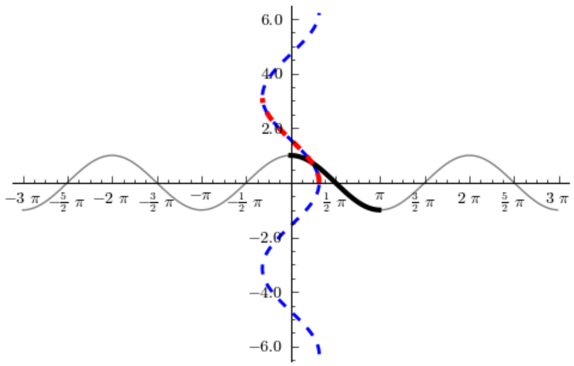
arcsin(x) is the (circle one) angle / number between and whose ...

$y = \arcsin(x)$ means:

Alternative Notation:

Warning:

Inverse Cosine Function



Restricted $\cos(x)$ **has** **Domain:** **Range:**

arccos(x) has Domain: Range:

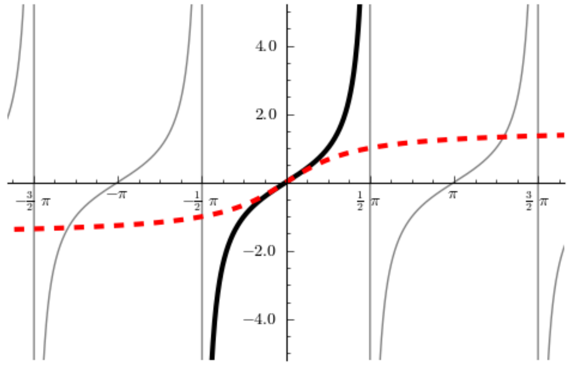
arccos(x) is the (circle one) angle / number between and whose ...

$y = \arccos(x)$ means:

Alternative Notation:

Warning:

Inverse Tangent Function



Restricted $\tan(x)$ **has** **Domain:** **Range:**

$\arctan(x)$ has Domain: Range:

$\arctan(x)$ is the (circle one) angle / number between and whose ...

$y = \arctan(x)$ means:

Alternative Notation:

Warning:

END OF VIDEO

Example. 1. Find an angle θ with $0 \leq \theta \leq 360°$ that has the same cosine as the angle $42°$.

2. Find an angle θ with $0 \leq \theta \leq 2\pi$ that has the same cosine as the angle $\dfrac{9\pi}{8}$.

Example.

1. Find an angle θ with $0 \leq \theta \leq 360^\circ$ that has the same sine as the angle 200°.

2. Find an angle θ with $0 \leq \theta \leq 2\pi$ that has the same sine as the angle $\frac{5\pi}{7}$.

Example. Find the exact value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Example. Find the exact value of $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

A. $-\frac{2}{\sqrt{3}}$

B. $-\frac{\pi}{6}$

C. $\frac{2\pi}{3}$

D. $\frac{5\pi}{6}$

E. $\frac{7\pi}{6}$

Example. Find the exact value of $\tan^{-1}\left(-\sqrt{3}\right)$

A. $-\dfrac{\pi}{3}$

B. $-\dfrac{\pi}{6}$

C. $\dfrac{\pi}{6}$

D. $\dfrac{\pi}{3}$

E. $\dfrac{2\pi}{3}$

F. $\dfrac{5\pi}{6}$

Example. Find the exact value(s) of x if $-4\tan^{-1}(x) = \pi$. OMIT THIS NEXT TIME

NEXT TIME ADD THESE:

Example. Find the exact value of

(a) $\arccos(\cos(0.2))$

(b) $\arccos(\cos(-0.3))$

(c) $\cos(\arccos(-0.7))$

(d) $\cos(\arccos(1.6))$

Summary:

- $\arccos(\cos(a)) = a$ (and $\arcsin(\sin(a) = a)$) EXCEPT when ...

- $\cos(\arccos(a)) = a$ (and $\sin(\arcsin(a) = a)$) EXCEPT when ...

Example. Find the exact value of $\arcsin(\sin(\dfrac{9\pi}{7}))$.

A. $-\dfrac{9\pi}{7}$

B. $-\dfrac{2\pi}{7}$

C. $\dfrac{2\pi}{7}$

D. $\dfrac{9\pi}{7}$

E. Does not exist

Example. Find the exact value of $\sin(\arcsin(\dfrac{9\pi}{7}))$

A. $-\dfrac{9\pi}{7}$

B. $-\dfrac{2\pi}{7}$

C. $\dfrac{2\pi}{7}$

D. $\dfrac{9\pi}{7}$

E. Does not exist

Example. Find the exact value of $\sec(\tan^{-1}(-\frac{4}{7}))$

A. $-\dfrac{\sqrt{65}}{7}$

B. $-\dfrac{\sqrt{65}}{4}$

C. $\dfrac{\sqrt{65}}{7}$

D. $\dfrac{\sqrt{65}}{4}$

Example. Write the trigonometric expression $\sin(\arctan(w))$ as an algebraic expression in terms of w only, if w is positive: (NOTE CHANGED THIS TO BE SIN FOR TRICKIER POS/ NEG computation.

What if w is negative?

Example. Write the trigonometric expression $\csc(\arccos(w))$ as an algebraic expression in terms of w.

A. $-\dfrac{1}{\sqrt{1-w^2}}$

B. $-\dfrac{w}{\sqrt{1-w^2}}$

C. $\dfrac{1}{\sqrt{1-w^2}}$

D. $\dfrac{w}{\sqrt{1-w^2}}$

Trig Equations

After completing this section, students should be able to:

- Find the solutions in an interval for an equation that contains a single trig function (and is linear or quadratic in that trig function) like $\tan(x) = -\sqrt{3}$ or $2\sin^2(\theta) + 3\sin(\theta) = -1$ in the interval $[0, 2\pi)$.
- Find ALL solutions to an equation that contains a single trig function (and is linear or quadratic in that trig function) like $\tan(x) = -\sqrt{3}$ or $2\sin^2(\theta) + 3\sin(\theta) = -1$
- Find solutions for a trig equation that contains two trig functions, by using a Pythagorean identity to convert sin to cos or tan to sec, etc. For example, $4\sin^2(x) = 5 - 4\cos(x)$
- Find solutions to a trig equation involving an angle multiplied by a constant and / or summed with a constant. E.g. $\sin(2x. + \frac{\pi}{3}) + 1 = 0$
- Use a calculator to find all solutions to an equation when the solution angles are not special angles on the unit circle. E.g. $\cos(x) = 0.6$ or $2\sin^2(t) + 3\sin(t) = 1$.

Example. For the equation $2\cos(x) + 1 = 0$,

(a) Find the solutions in the interval $[0, 2\pi)$.

(b) Give a general formula for ALL solutions (not just those in the interval $[0, 2\pi)$.

Example. For the equation $2\tan(x) = \sqrt{3} - \tan(x)$

(a) Find the solutions in the interval $[0, 2\pi)$.

(b) Give a general formula for ALL solutions.

END OF VIDEO

Example. For the equation $\tan(x) = -\sqrt{3}$.

(a) Find all solutions in the interval $[0, 2\pi)$.

(b) Give a general formula for ALL solutions.

Example. For the equation $\sin(x) = -\sqrt{3} - \sin(x)$

(a) Find all solutions in the interval $[0, 2\pi)$.

(b) Give a general formula for ALL solutions.

Example. Find all solutions to the equation $2\cos^2(\theta) + 7\cos(\theta) = 4$

Example. Solve the equation $2\cos^2(x) = 3 + 3\sin(x)$

Example. Solve the equation on the interval $[0, 2\pi)$:

$$\sec^2(x) - \tan(x) = 1$$

Variation: what if the argument is more complicated?
Example. For the equation $2\sin(3x) - \sqrt{2} - 0$

(a) Find the general solution set.

(b) Find the solutions on the interval $[0, 2\pi)$

Extra Example. Find all solutions to $\sin\left(\frac{x}{2} + \frac{\pi}{3}\right) = 1$

Variation: what if the angles that are needed aren't special angles on the unit circle?
Example. For the equation $2\cos(t) = 1 - \cos(t)$

(a) Find all solutions in the interval $[0, 2\pi)$

(b) Find all solutions.

Example. For the equation $4\sin(x) - 1 = 2$

(a) Find all solutions in the interval $[0, 2\pi)$

(b) Find all solutions.

Example. For the equation $\tan(\theta) = 4$

(a) Find all solutions in the interval $[0, 2\pi)$

(b) Find all solutions.

Summary:

(a) To find solutions to the equation $\cos(t) = M$

(b) To find solutions to the equation $\sin(t) = N$

(c) To find solutions to the equation $\tan(t) = P$

Example. Find all solutions to the equation $\sin^2(x) + 3\cos(x) = -2$

Extra Example. Find all solutions to the equation $\sin^2(x) + 1 = 5\sin(x)$

Trig Identities

After completing this section, students should be able to:

- Build evidence to decide if an equation is a trig identity by plugging in numbers or graphing both sides.
- Prove that an equation is NOT a trig identity by plugging in numbers for which it does not hold.
- Prove that an equation IS a trig identity by rewriting one side of the equation using algebra, definitions of trig functions, and/ or the Pythagorean identity, until it matches the other side of the equation.

Example. Find solutions to the following equations:

a) $x^2 - 6x = 7$

b) $x^2 - 6x = 7 + (x - 7)(x + 1)$

Definition. The second equation is called an *identity* because ...

Example. Decide which of the following equations are identities.

a) $\sin(2t) = 2\sin(t)$

b) $\cos(\theta + \pi) = -\cos(\theta)$

c) $\sec(x) - \sin(x)\tan(x) = \cos(x)$

The following identities are called the Pythagorean identities.

1. $\cos^2(\theta) + \sin^2(\theta) = 1$
2. $\tan^2(\theta) + 1 = \sec^2(\theta)$
3. $\cot^2(\theta) + 1 = \csc^2(\theta)$

Why do they hold?

END OF VIDEOS

Question. What makes an equation an *identity*?

Example. Which of the following equations are identities?

(a) $(1 - \sin(x))(1 + \sin(x)) = \cos^2(x)$

(b) $\cot(x)\sin(x) = \tan(x)$

Example. Prove the identity

$$\frac{\cos^2(x)}{1-\sin(x)} = \frac{\csc(x)+1}{\csc(x)}$$

Example. Prove the identity.

$(\csc^2(x) - 1)\sin(x) = \cot(x)\cos(x)$

Example. Prove the identity.

$$\frac{1}{\csc(x) + 1} + \frac{1}{\sin(x) + 1} = 1$$

Extra Example. Prove the following trig identity.

$$\frac{\cot(x)}{\csc(x)+1} = \frac{\csc(x)-1}{\cot(x)}$$

Angle Sum and Difference Formulas

After completing this section, students should be able to:

- Use the sum and difference formulas to calculate the sin and cos of angles like $\frac{\pi}{12}$ that can be written as the sum or difference of two special angles on the unit circle.
- Find $\sin(a+b)$ and $\cos(a+b)$ form information about $\sin(a)$ and $\sin(b)$ (or $\cos(a)$ and $\cos(b)$, etc.) and the quadrant for a and for b.
- Simplify expressions like $\cos(25°)\sin(5°) + \sin(25°)\cos(5°)$.
- Simplify expressions like $\cos(\sin^{-1}(u) - \sin^{-1}(v))$.
- Use the sum and difference formulas to prove trig identities like $\sin(\frac{\pi}{2}+x) = \cos(x)$.

Question. Is it true that $\sin(A + B) = \sin(A) + \sin(B)$?

$\sin(A + B) =$

$\cos(A + B) =$

$\sin(A - B) =$

$\cos(A - B) =$

Example. Find an exact value for $\sin(105°)$

Example. If $\cos(v) = 0.9$ and $\cos(w) = 0.7$, find $\cos(v + w)$. Assume v and w are in the first quadrant.

END OF VIDEO

Question. True or False: $\cos(A + B) = \cos(A) + \cos(B)$

$\sin(A + B) =$

$\cos(A + B) =$

$\sin(A - B) =$

$\cos(A - B) =$

Example. Find an exact value for $\sin\left(\frac{\pi}{12}\right)$

Example. Prove the trig identity $\cos(\theta + \pi) = -\cos(\theta)$.

Example. If $\sin(A) = \dfrac{3}{5}$ and $\cos(B) = \dfrac{1}{3}$, find $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$.

Example. Find an expression for $\cos(\sin^{-1}(u) - \sin^{-1}(v))$.

Note for next time: could combine this problem and the next one into one problem by doing something like $\cos(\sin^{-1}(u) - \frac{5\pi}{6})$. ALSO be sure to check Gracie's MyLabMath assignments in class for good problems.

Extra Example. If $\cot\theta = -7$ and $\sec\theta < 0$, find the exact value of $\sin(\theta + \frac{5\pi}{6})$.

Double Angle and Half Angle Formulas

After completing this section, students should be able to:

- Show how the double angle formulas follow from the angle sum formula.
- Show how the half angle formulas follow from the double angle formulas for cos.
- Use the double angle formulas to solve equations like $\cos(2x) + \cos(x) = 0$
- Use the double angle formulas to calculate expressions like $\sin(2x)$ given information about $\sin(x)$ or $\cos(x)$.
- Use the double angle formulas to rewrite expressions like $\cos(2\sin^{-1}(w))$.
- Use the double angle formula to prove identities like $\dfrac{2\tan(x)}{1+\tan^2(x)} = \sin(2x)$
- Use the half angle formula to compute exact values like $\sin\left(\frac{7\pi}{12}\right)$
- Use the half angle formula to compute values like $\sin(\frac{A}{2})$ given information about $\sin(A)$ or $\cos(A)$.

Question. True or False: $\sin(2\theta) = 2\sin(\theta)$

Double Angle Formulas:

$\sin(2\theta) =$

$\cos(2\theta) =$

Example. Find $\cos(2\theta)$ if $\cos(\theta) = -\frac{1}{\sqrt{10}}$ and θ terminates in quadrant III.

Example. Solve the equation $2\cos(x) + \sin(2x) = 0$.

Half Angle Formulas:

$$\cos\left(\frac{\theta}{2}\right) =$$

$$\sin\left(\frac{\theta}{2}\right) =$$

Example. Suppose that $\sin(\theta) = \frac{4}{5}$ and $\frac{\pi}{2} < \theta < \pi$. Find the exact values of $\cos\left(\frac{\theta}{2}\right)$ and $\sin\left(\frac{\theta}{2}\right)$.

END OF VIDEOS

Question. Which of the following are trig identities (list the double angle and half angle formulas ... maybe leave out the plus and minus signs for the half angles?) this will be a way to review. Or could do the true false $\sin(2x) = .2\sin(x)$ routine or maybe $\sin(\frac{\theta}{2}) = \frac{\sin(\theta)}{2}$ since the other one was on the video.

Example. If $\cos(\theta) = -\dfrac{2}{5}$ and $\sin(\theta) > 0$, find $\sin(2\theta)$.

Example. Find an algebraic expression for $\cos(2\sin^{-1}(w))$.

Example. Solve on the interval $0 \le \theta < 2\pi$: $\cos(2\theta) = \sin(\theta)$

Example. Solve for all values of x $\sin(2x) + \sqrt{3}\cos(x) = 0$.

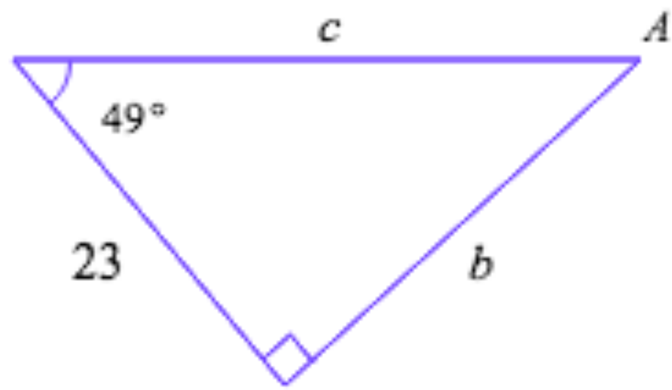
Example. Find the exact value of $\cos\left(\dfrac{\pi}{8}\right)$

Applications of Right Angle Trigonometry

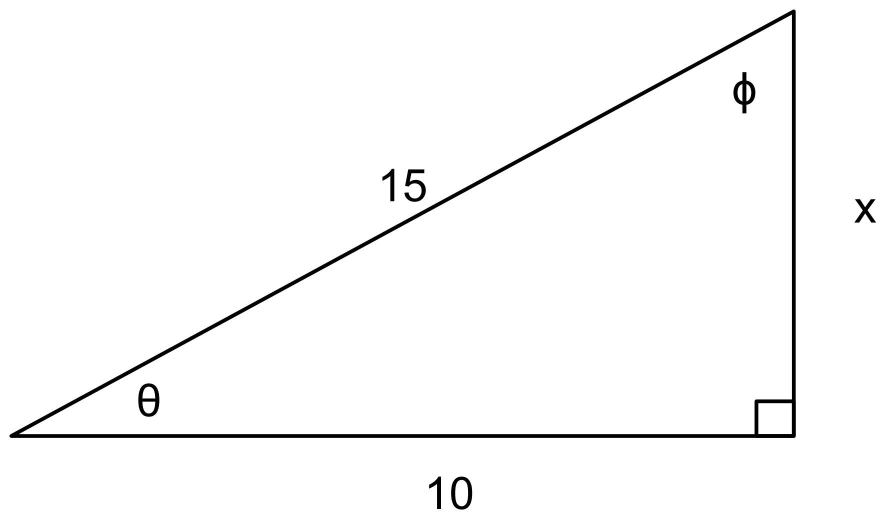
After completing this section, students should be able to:

- Solve a right triangle given information about a (non-right) angle and one side, or two sides.
- Solve for side lengths and angles fo right triangles in application problems (word problems).
- Use trig to find the area of a right triangle, given an angle and one side.
- Use trig to find the area of a triangle, given the lengths of two of the sides and the measure of the angle between them.

Example. Solve the right triangle.



Example. Solve the right triangle.



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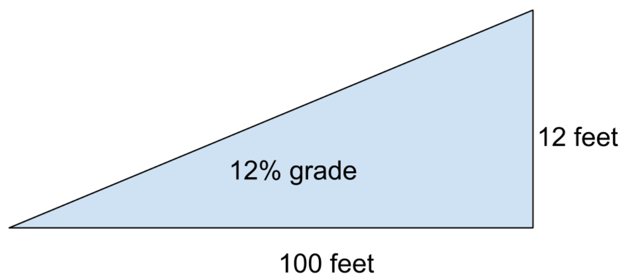
Example. A 12% grade means that the vertical rise is 12 feet per 100 feet of horizontal distance.



A. Suppose that a road climbs at an angle of 15°. What is its grade?

B. What is the angle of a road that has an 8% grade?

Extra Example. A security camera in a university lab is mounted on a wall 9 feet above the floor. What angle of depression should be used if the camera is to be directed to a spot 6 feet above the floor and 12 feet from the wall?

Extra Example. An airplane leaves from its runway, whose bearing is $N50°E$. After flying for 1 mile, the pilot requests permission to turn 90° and head toward the southeast. After the airplane goes 2 miles in this direction, what bearing should the control tower use to locate the plane?

Law of Cosines

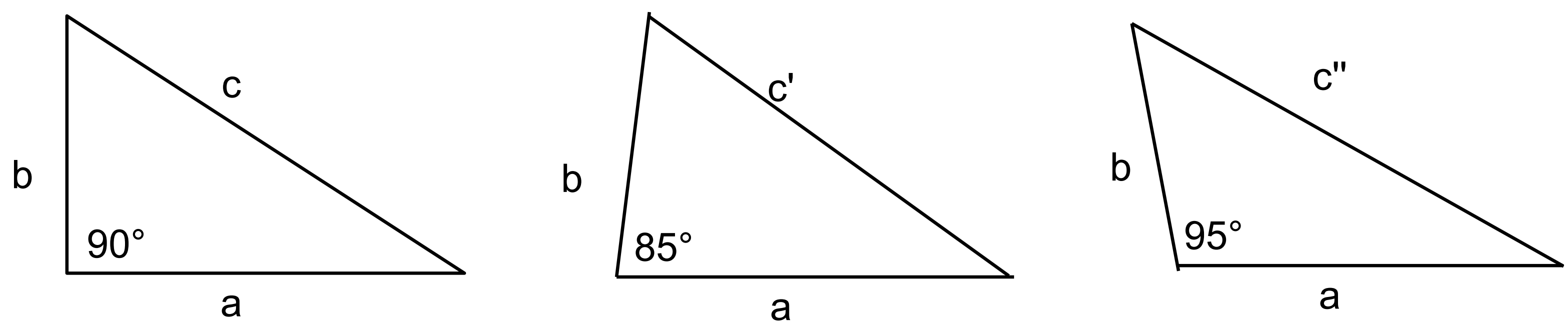
After completing this section, students should be able to:

- Use the Law of Cosines to solve a triangle, given the lengths of two sides and the measure of and angle between them (SAS).
- Use the Law of Cosines to solve a triangle, given the lengths of three sides (SSS).
- Use the Law of Cosines to find angles and lengths in a word problem.

The *Law of Cosines* is a generalization of the Pythagorean Theorem to triangles that are not right triangles. It says that for any triangle,

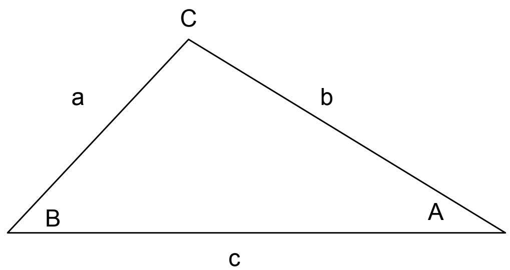
$$c^2 = a^2 + b^2 + \text{correction factor}$$

The correction factor depends on the angle opposite side c.



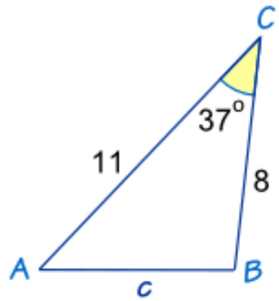
Theorem. *(The Law of Cosines) For any triangle with sides a, b, and c and angle C opposite side c,*

Notation conventions:

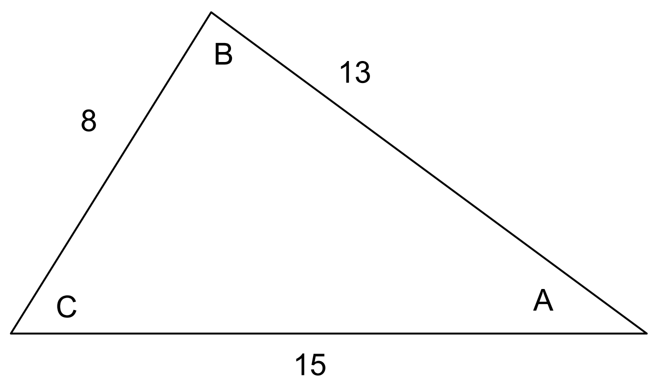


The Law of Cosines can be written in three forms:

Example. Find all side lengths and angles of this triangle.



Example. Find the angles of this triangle.



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Review. Which of the following equations gives the Law of Cosines?

A. $c^2 = a^2 - b^2 + 2bc\cos A$

B. $c^2 = a^2 + b^2 + 2ab\cos A$

C. $c^2 = a^2 + b^2 - 2ab\cos C$

D. $a^2 = b^2 + c^2 - 2bc\cos A$

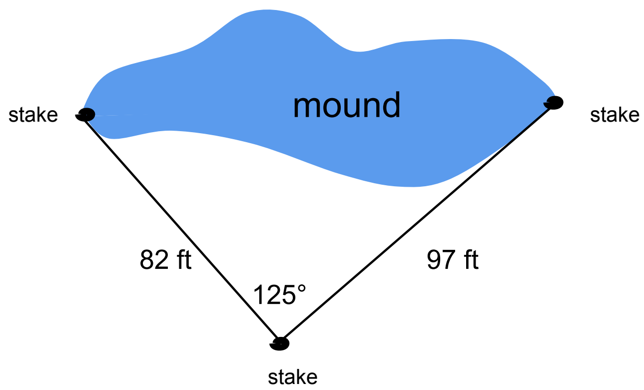
E. $b^2 = a^2 + c^2 - 2ac\cos B$

Question. How many variables does the Law of Cosines have in it?

Question. How many sides or angles do you need to know in order to use the Law of Cosines to solve for an additional side or an angle?

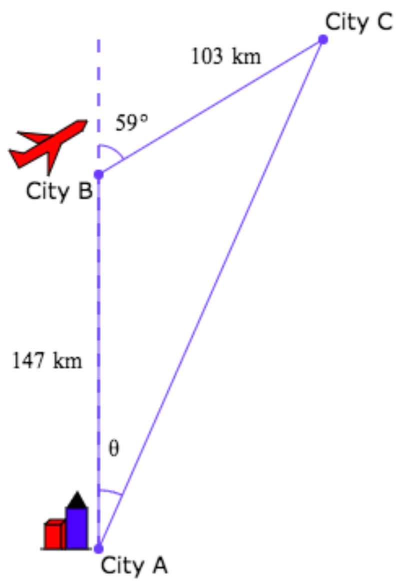
Example. Two kayaks leave the shore at the same time, traveling on courses that have an angle of $85°$ between them. If both kayaks travel at 6 mph, how far apart are they after 30 minutes? Answer to the nearest mile.

Extra Example. To estimate the width of the archaeological mound, archaeologists place two stakes on opposite ends of the widest point. They set a third stake off to the side, and connect ropes between the stakes as shown. Find the width of the mound.

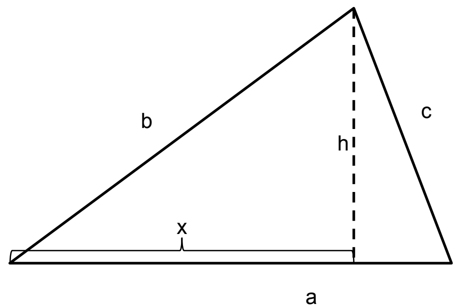


Example. After the hurricane, the small tree in my neighbor's yard was leaning. To keep it from falling, we nailed a 6-foot strap into the ground 4 feet from the base of the tree. We attached the strap to the tree 3.5 feet up the tree trunk. How far from vertical was the tree leaning?

Example. An airplane leaves City A and flies 147 km due north to City B. It then turns through an angle of 59° and flies 103 km to City C. What angle θ with respect to due north could the pilot have used to fly directly from City A to City C?



Proof. (Law of Cosines) Give a reason for each of the following statements to justify the proof of the Law of Cosines.



Statement	Reason
$x^2 + h^2 = b^2$ and $(a - x)^2 + h^2 = c^2$	
$h^2 = b^2 - x^2$ and $h^2 = c^2 - (a - x)^2$	
$b^2 - x^2 = c^2 - (a - x)^2$	
$c^2 = b^2 - x^2 + (a - x)^2$	
$c^2 = a^2 + b^2 - 2ax$	
$\frac{x}{b} = \cos(C)$	
$c^2 = a^2 + b^2 - 2ab\cos(C)$	

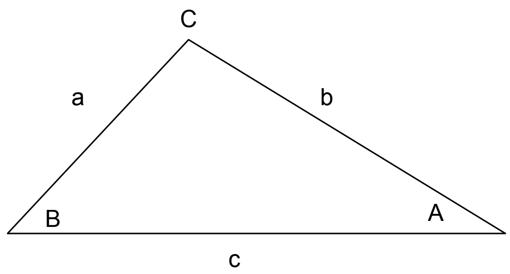
Note: this proof works for a triangle in which all angles are acute ($\leq 90°$). A slightly different argument is needed to handle obtuse angles ($> 90°$).

Law of Sines

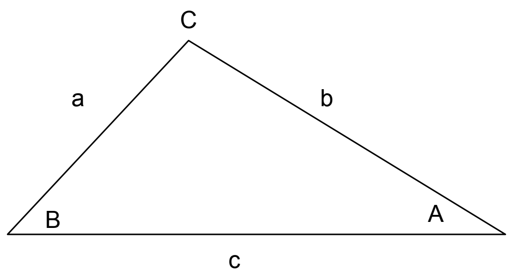
After completing this section, students should be able to:

- Use the Law of Sines to solve a triangle, given the length of one side and the measure of two angles (ASA).
- Explain why knowing the length of two sides and an angle NOT between them (SSA) does not always determine the shape of the triangle completely (the ambiguous case SSA).
- Use the Law of Sines to solve a triangle, given the length of two sides and one angle opposite to one of these sides, or give two options when possible in the ambiguous case SSA.
- Use the Law of Sines to show that a given triangle with two specified side lengths and one measured angle not between them (SSA) is impossible (because the sine of an angle would have to be greater than 1).
- Use the Law of Sines to find angles and lengths in a word problem.

Theorem. *(Law of Sines) For a triangle with angles A, B, and C and opposite sides a, b, and c,*



Example. Suppose $A = 55°$, $C = 67°$, and $b = 20$. Solve the triangle.



Caution: When using the Law of Sines to find angles, the Law of Sines only gives us the sine of an angle, not the angle, and there are two angles beteween $0°$ and $180°$ that have the same sine.

For example, if use the Law of Sines to find that $\sin(A) = 1/2$, we don't know whether $A = 30°$ or $A = 150°$.

Sometimes we can use clues from the problem to eliminate one solution as impossible, but sometimes both solutions are possible.

Example. (an ambiguous case) Suppose that $a = 8$ and $b = 7$, and $B = 40°$. Solve the triangle, finding all possible solutions.

Example. (the amgibuous case revisited) Solve the same triangle with $a = 8$ and $b = 7$, and $B = 40°$ using the Law of Cosines. Do we still get two solutions?

Review. Which of the following equations is the Law of Sines?

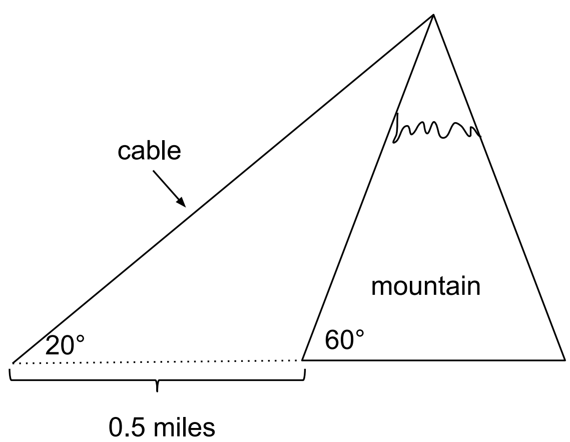
A. $a\sin(A) = b\sin(B) = c\sin(C)$

B. $\dfrac{\sin(A)}{a} = \dfrac{\sin(B)}{b} = \dfrac{\sin(C)}{c}$

C. $\dfrac{\sin(a)}{A} = \dfrac{\sin(b)}{B} = \dfrac{\sin(c)}{C}$

Question. You may remember from geometry that in a triangle, bigger sides are opposite bigger angles. How does this fact follows from the Law of Sines? In other words, if angle $A >$ angle B, why does it follow that side $a >$ side b?

Example. An areal tram starts at a point $\frac{1}{2}$ mile fom the base of a mountain whose face has a $60°$ angle. The tram ascends at an angle of $20°$. What is the length of the cable?



Extra Example. A pole tilts westward toward the sun at an angle of 8° from vertical. It casts a 22-foot shadow to the east. The angle of elevation from the tip of the shadow to the top of the pole is 43° To the nearest tenth foot, how long is the pole?

Example. Princess Island lies due west of Devil's Island. A boat leaves Devil's Island and travels for a 2 hours at 24 miles an hour at a bearing of $N78°W$. The captain then corrects course and continues at the same speed, reaching Princess Island an hour and 15 minutes later. What is the distance between Princess Island and Devil's Island and through what angle did the captain have to turn to correct course?

Hint: two solutions are possible, but one is more plausible.

Extra Example. Two tracking stations are on an East-West line 110 miles apart. A forest fire is located on a bearing of $N42°E$ from the western station at A and a bearing of $N15°E$ from the eastern station of B. To the nearest tenth mile, how far is the fire from the western station?

Question. In which of the following situations can you use the Law of Cosines to solve the triangle? In which can you use the Law of Sines?

A. Know three sides (SSS)

B. Know two sides and the angle between them (SAS)

C. Know two sides and an angle that is not between them (SSA)

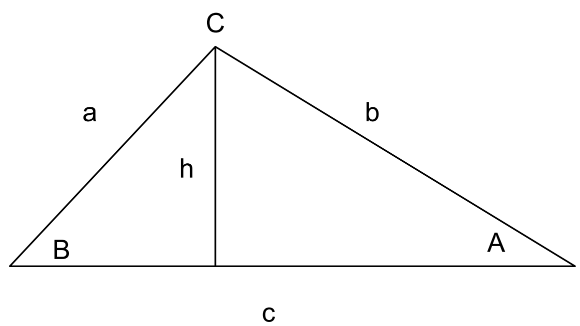
D. Know two angles and the side between them (ASA)

E. Know two angle and a side that is not between them (AAS)

F. Know three angles (AAA)

Question. Which of these can have multiple solutions?

Proof. (Law of Sines) Give a reason for each of the following statements to justify the proof of one part of the Law of Sines.



Statement	Reason
$\sin(B) = \frac{h}{a}$ and $\sin(A) = \frac{h}{b}$	
$h = a\sin(B)$ and $h = b\sin(A)$	
$a\sin(B) = b\sin(A)$	
$\frac{\sin(A)}{a} = \frac{sin(B)}{b}$	

Note: this proof works for a triangle in which all angles are acute (≤ 90°). A slightly different argument is needed to handle obtuse angles (> 90°).

Note. It's a pain that the Law of Sines can give two answers when you use it to find angles. Even worse, sometimes one of the two answers is extraneous. When possible, I recommend using the Law of Cosines to find angles, instead. Of course, if you already know two angles, you can find the third just by subtracting from 180°.