

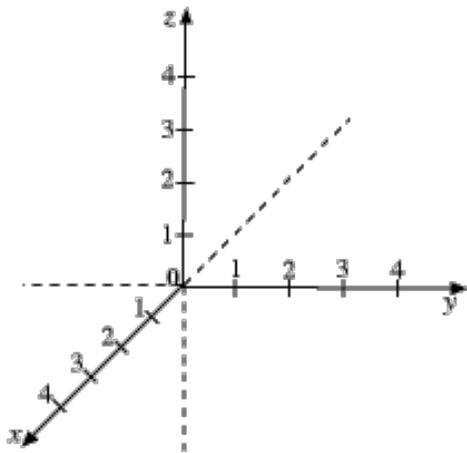
## §13.6 - Cylinders and Quadric Surfaces

After completing this section, you should be able to:

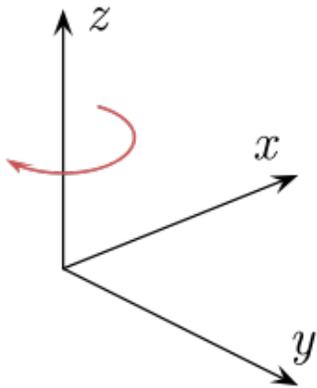
- Recognize right handed and left handed coordinate systems
- Determine whether an equation describes a plane, cylinder, sphere, or other standard shape in 3-dimensional space
- Describe the regions of 3-dimensional space defined by inequalities
- Complete the square to rewrite equations and inequalities involving 3 variables in more standard form
- Match equations of quadric surfaces with their graphs
- Identify traces of a surface based on the equation of the surface.

**Definition.**  $\mathbb{R}^3$  means

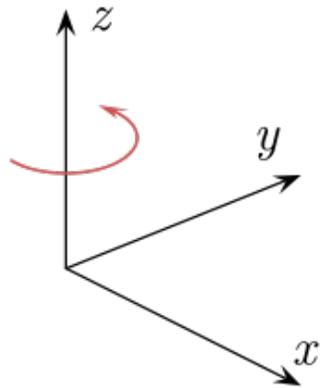
**Example.** Sketch the point  $(2, -3, 1)$  on the coordinate axes below.



By convention, we graph points in  $\mathbb{R}^3$  using a right-handed coordinate system. *Right-handed* means:

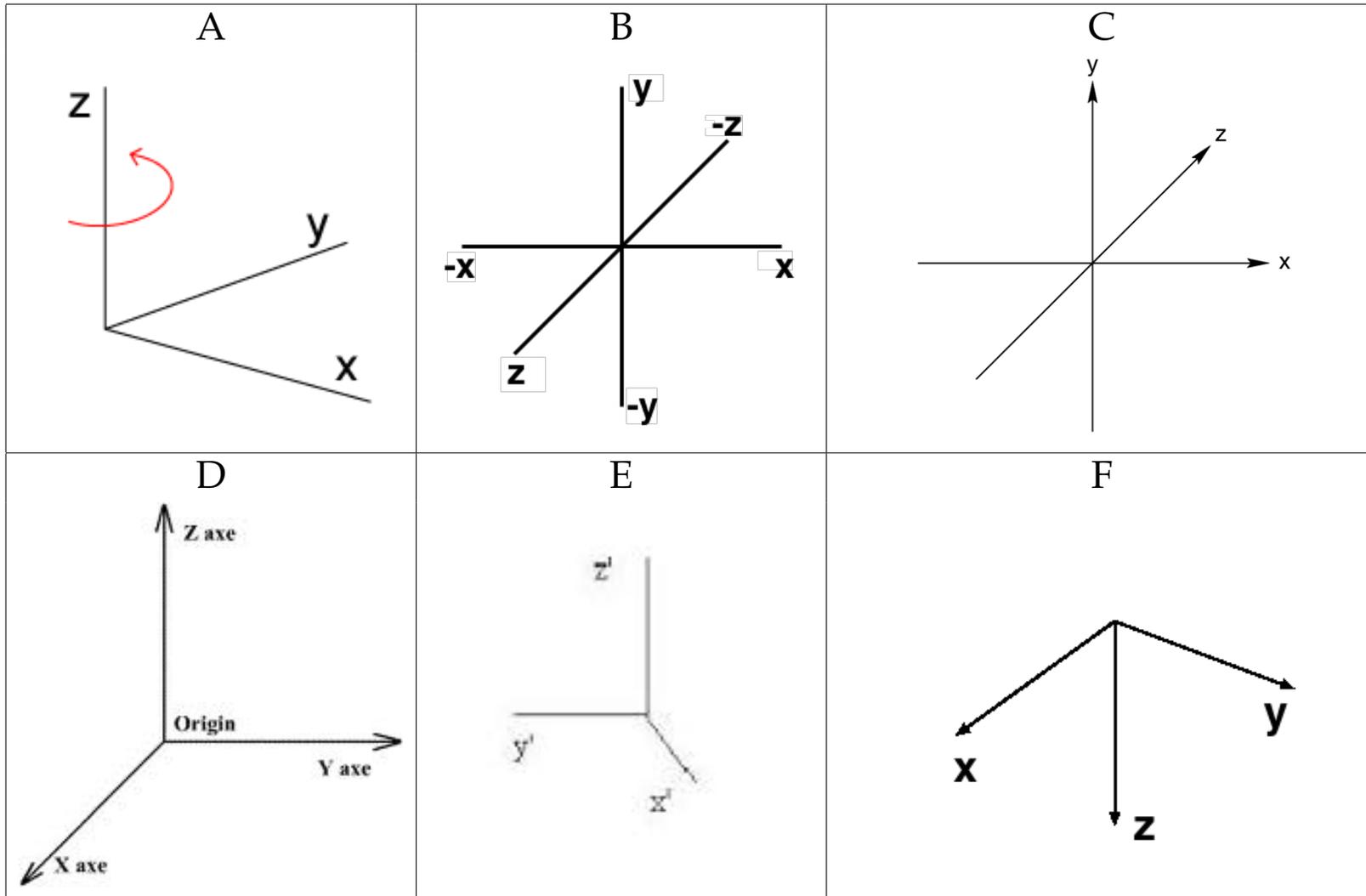


Left-handed coordinate system



Right-handed coordinate system

Identify the right-handed coordinate systems.



**Question.** For two points  $P_1 = (x_1, y_1, z_1)$  and  $P_2 = (x_2, y_2, z_2)$  in  $\mathbb{R}^3$ , what is the distance between  $P_1$  and  $P_2$ ?

**Question.** What is the equation of a sphere of radius  $r$  centered at the point  $(h, k, l)$ ?

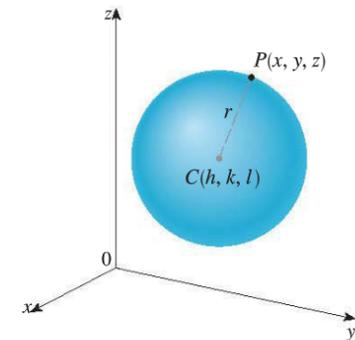
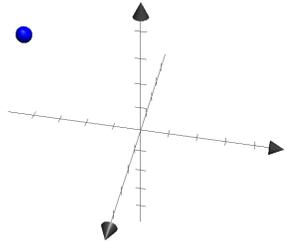


FIGURE 12

**Example.** Find the distance from  $(4, -2, 6)$  to each of the following:



- a. The point  $(9, -1, -4)$
- b. The  $xy$ -plane (where  $z = 0$ )
- c. The  $xz$ -plane
- d. The  $x$ -axis

**Example.** Describe the region and draw a picture.

a.  $x < 2$

b.  $x^2 + z^2 \leq 9$

c.  $x^2 + y^2 + z^2 > 2z$

**Extra Example.** Consider the points  $P$  such that the distance from  $P$  to  $A(6, 2, -2)$  is half the distance from  $P$  to  $B(-1, 5, 3)$ . Show that the set of all such points is a sphere, and find its center and radius.

**Note.** In  $\mathbb{R}^2$ , what shapes do the following equations represent?

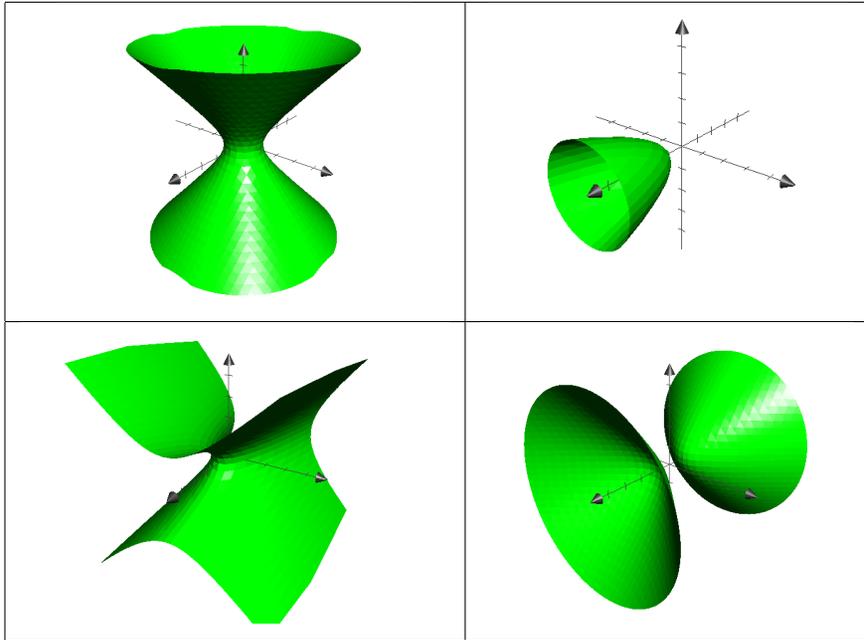
- $x^2 + y^2 = 25$

- $x + y^2 = 25$

- $x^2 - y^2 = 25$

- $-x^2 - y^2 = 25$

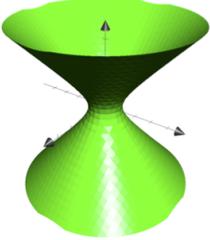
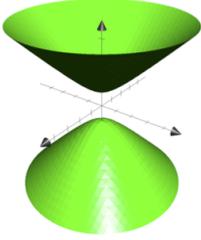
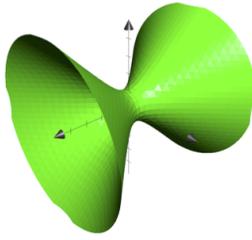
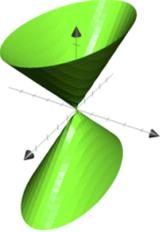
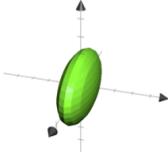
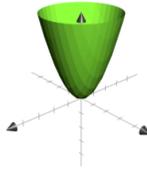
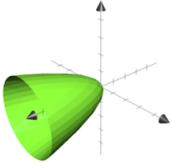
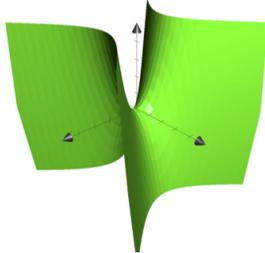
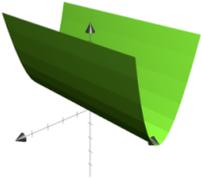
**Example.** Without using graphing software, identify the graph of  $x + y^2 - z^2 = 1$ ?



**Note.** It can be helpful to consider *intercepts* and *traces*.

**Definition.** A **trace** of a surface is the set of point where ....

Match the equation to the graph using only your brain (no graphing software).

<p>A.</p> 	<p>B.</p> 	<p>C.</p> 
<p>D.</p> 	<p>E.</p> 	<p>F.</p> 
<p>G.</p> 	<p>H.</p> 	<p>I.</p> 

1. $x^2 + z^2 = 4$	2. $36x^2 + 4y^2 + 9z^2 = 36$	3. $-x^2 - y^2 + z = 0$	4. $-x^2 + y^2 + z = 0$
5. $x - y^2 - z^2 = 0$	6. $-x^2 - 4y^2 + z^2 = 0$	7. $x^2 + y^2 - z^2 = 1$	8. $-x^2 - y^2 + z^2 = 1$
9. $-x^2 + y^2 + z^2 = 1$	10. $-x^2 + z = 0$		

## §13.1 and 13.2 Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

After completing this section, you should be able to:

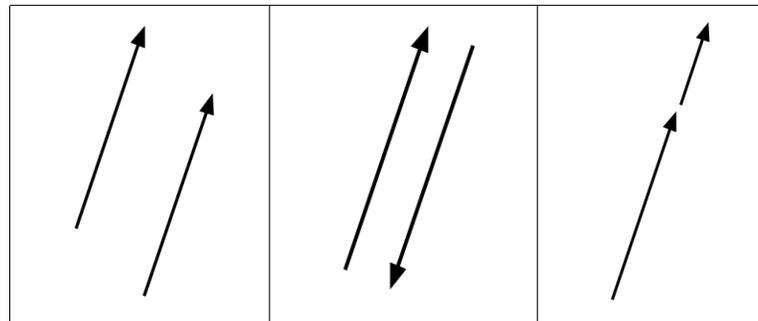
- Represent vectors with arrows, components and in terms of  $\vec{i}$ ,  $\vec{j}$ ,  $\vec{k}$ .
- Add and subtract vectors
- Find the length of a vector
- Rescale vectors to find other vectors in same direction
- Use vectors to solve problems from physics and geometry

**Definition.** A *vector* is a quantity with direction and magnitude (length).

Vectors are usually drawn with arrows.

Two vectors are considered to be the same if:

**Example.** Which pairs of vectors are equivalent?



**Definition.** A *scalar* is another word for \_\_\_\_\_ .

A scalar does not have a direction, in contrast to a vector.

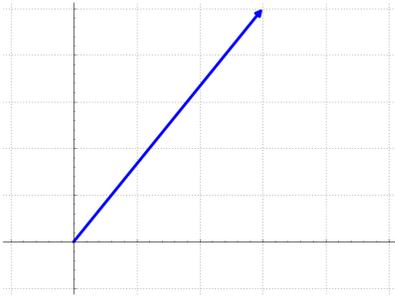
Vector addition: Draw  $\vec{a} + \vec{b}$

Vector subtraction: Draw  $\vec{a} - \vec{b}$

Multiplication of scalars and vectors: Draw  $2\vec{a}$  and  $-3\vec{a}$

## Vector Components

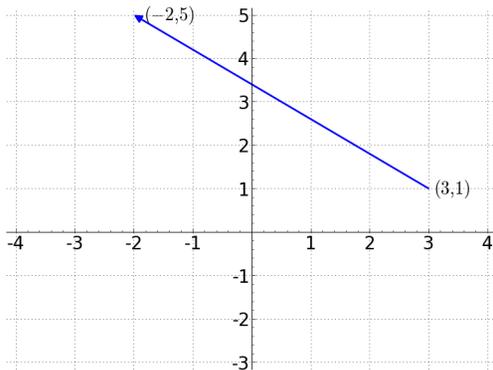
If we place the initial point of a vector  $\vec{a}$  at *the origin*, then the vector can be described by



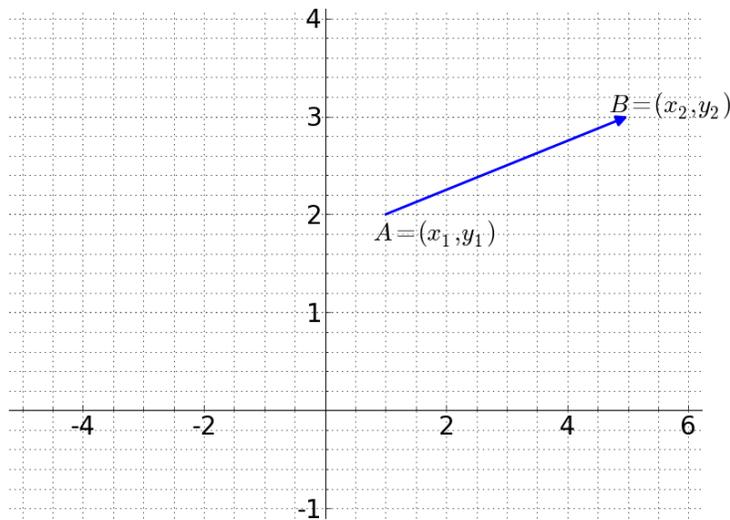
**Definition.** The *components* of a vector are

**Note.** Vectors in 3-dimensional space can be described in terms of three components.

**Question.** What are the components of the vector  $\vec{AB}$  that starts at a point  $A = (3, 1)$  and ends at a point  $B = (-2, 5)$ ?



**Note.** In general, the components of the vector that starts at a point  $A = (x_1, y_1)$  and ends at a point  $B = (x_2, y_2)$  are:



**Definition.** Given two vectors  $\vec{a} = \langle a_1, a_2 \rangle$  and  $\vec{b} = \langle b_1, b_2 \rangle$ , and a scalar  $c$

$$\vec{a} + \vec{b} =$$

$$\vec{a} - \vec{b} =$$

$$c\vec{a} =$$

Similarly, for three-dimensional vectors  $\vec{a} = \langle a_1, a_2, a_3 \rangle$  and  $\vec{b} = \langle b_1, b_2, b_3 \rangle$ ,

$$\vec{a} + \vec{b} =$$

$$\vec{a} - \vec{b} =$$

$$c\vec{a} =$$

**Example.**  $\langle 1, 2, 3 \rangle + \langle 5, 7, 12 \rangle =$

**Question.** We have defined vector addition, subtraction, and scalar multiplication twice.

Why are these definitions equivalent?

1. Addition is commutative	
2. Addition is associative	
3. Additive identity (the zero vector)	
4. Additive inverses	
5. Distributive property	
6. Distributive property	
7. Associativity of scalar multiplication	
8. Multiplicative identity	

**Definition.** The *length* of a vector  $\vec{w} = \langle w_1, w_2, w_3 \rangle$  is

$$\|\vec{w}\| =$$

**Note.** The length of a vector is also called:

**Definition.** A *unit vector* is a vector ...

**Question.** How can we rescale any vector to make it a unit vector? (Rescale means multiply by a scalar.)

**Note.** Rescaling a vector to make it a unit vector is also called ...

**Example.** Find a unit vector that has the same direction as  $\langle 5, 1, 3 \rangle$ .

**Definition.** In 2 dimensions, the *standard basis vectors* are

**Definition.** In 3 dimensions, the *standard basis vectors* are

**Note.** Vectors can always be written in terms of

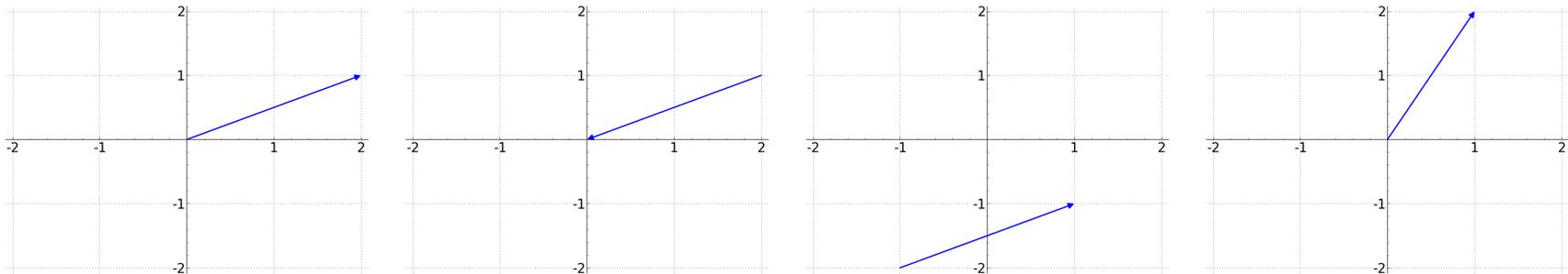
**Example.** Write  $\langle 3, -2, 7 \rangle$  as a sum of multiples of standard basis vectors.

END OF VIDEOS

**Review.** Vectors can be represented with

- 
- 
- 

**Example.** Which arrow(s) represent(s) the vector  $\langle 2, 1 \rangle$ ?



**Example.** Write  $\langle 2, 1 \rangle$  in terms of the standard basis vectors  $\vec{i}$  and  $\vec{j}$ .

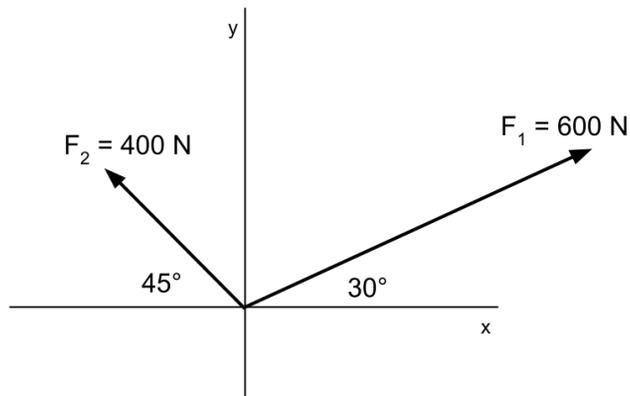
**Example.** Find a unit vector in the direction of  $\langle 2, 1 \rangle$ .

**Example.** Find the unit vectors that are parallel to the tangent line to  $y = x^3$  at  $(2, 8)$ .

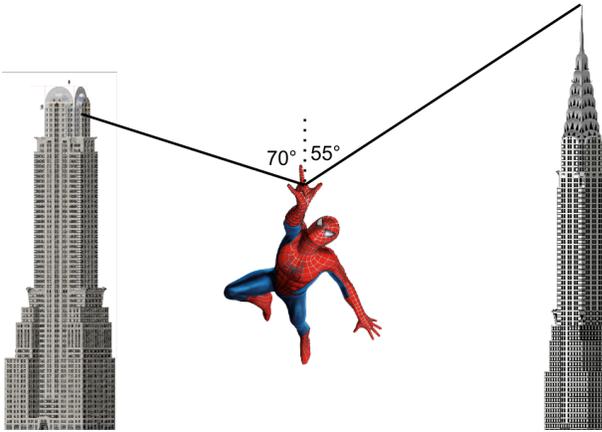
**Example.** The 2-d unit vector  $\vec{u}$  makes an angle of  $\frac{\pi}{3}$  with the positive x-axis. Find the components of  $\vec{u}$  and write  $\vec{u}$  in terms of  $\vec{i}$  and  $\vec{j}$ .

**Example.** The 2-d vector  $\vec{v}$  has magnitude 9 and makes an angle of  $\frac{5\pi}{6}$  with the positive x-axis. Find the components of  $\vec{v}$  and write  $\vec{v}$  in terms of  $\vec{i}$  and  $\vec{j}$ .

**Example.** Two ropes are attached to Silent Sam and pulled with the forces shown. Find the magnitude of the resultant force and the angle it makes with the positive x-axis.



**Extra Example.** Spiderman is suspended from two strands of spider silk as shown. Find the tension in each strand of spider silk. (You will need some additional information.)



## §13.3 Dot Product

After completing this section, students should be able to:

- Compute the dot product of two vectors from the vector components or from the magnitudes and the angle between them
- Use the dot product to find the angle between two vectors
- Use the dot product to determine if two vectors are perpendicular
- Find the scalar and vector projections of one vector onto another
- Compute the work done by a constant force moving an object in a direction that is at an angle to the direction of the force

**Definition.** If  $\vec{a} = \langle a_1, a_2, a_3 \rangle$  and  $\vec{b} = \langle b_1, b_2, b_3 \rangle$ , then the dot product of  $\vec{a}$  and  $\vec{b}$  is given by

$$\vec{a} \cdot \vec{b} =$$

**Example.** Find the dot product of  $\langle 4, 2, -1 \rangle$  and  $\langle 7, 0, 5 \rangle$ .

**Question.** Is  $\vec{a} \cdot \vec{b}$  a vector or a scalar?

**Note.** Dot product satisfies the following properties:

1. Commutative Property:

2. Distributive Property:

3. Associative Property:

4. Multiplication by  $\vec{0}$ :

**Question.** What do you get when you take the dot product of a vector with itself?

**Definition.** Dot product can also be defined in terms of magnitudes and angles:

**Example.** Suppose  $\vec{v}$  and  $\vec{w}$  meet at a  $45^\circ$  angle and  $\|\vec{v}\| = 4$  and  $\|\vec{w}\| = 6$ . Find  $\vec{v} \circ \vec{w}$ .

**Example.** Suppose  $\vec{a} \circ \vec{b} = 20$  and  $\|\vec{a}\| = 10$  and  $\|\vec{b}\| = 4$ . What is the angle between  $\vec{a}$  and  $\vec{b}$ ?

**Note.** If  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$ , then

$$\cos(\theta) =$$

**Example.** Find the angle between the vectors  $\langle 3, 1, 2 \rangle$  and  $\langle 4, 6, 1 \rangle$ .

**Question.** We have two definitions of dot product:

- 
- 

Are they equivalent?

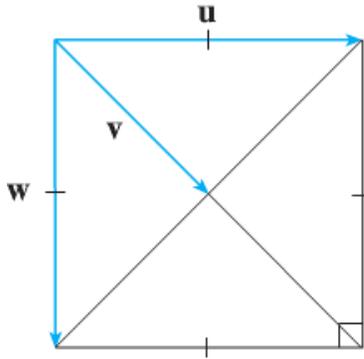
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**Review.** For  $\vec{a} = \langle a_1, a_2, a_3 \rangle$  and  $\vec{b} = \langle b_1, b_2, b_3 \rangle$ , what are the two ways of defining  $\vec{a} \circ \vec{b}$ ?

- 

-

**Example.** If  $\vec{u}$  is a unit vector, find  $\vec{u} \circ \vec{v}$  and  $\vec{u} \circ \vec{w}$ .



**Question.** Assume  $\vec{a} \neq \vec{0}$  and  $\vec{b} \neq \vec{0}$ ? What can be said about the angle between  $\vec{a}$  and  $\vec{b}$ , if

- $\vec{a} \circ \vec{b} = 0$

- $\vec{a} \circ \vec{b} > 0$

- $\vec{a} \circ \vec{b} < 0$

**Question.** Is it possible to find the exact angle between two vectors if all we know is the dot product and the magnitudes of the vectors?

**Example.** Find the (smaller) angle between the lines. Round your answer to the nearest degree.

$$5x - y = 5, 9x + y = 6$$

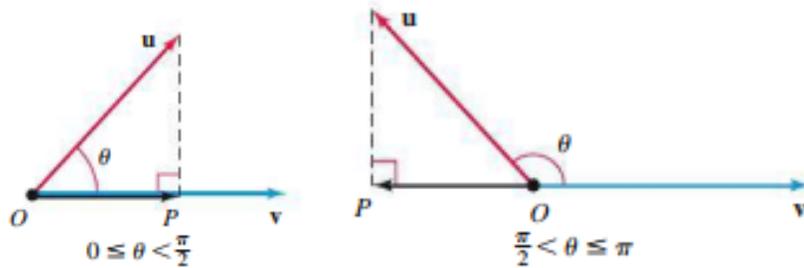
**Example.** True or False:

(a)  $\vec{i} \circ \vec{j} = \vec{j} \circ \vec{k} = \vec{k} \circ \vec{i} = 0$

(b)  $\vec{i} \circ \vec{i} = \vec{j} \circ \vec{j} = \vec{k} \circ \vec{k} = 1$

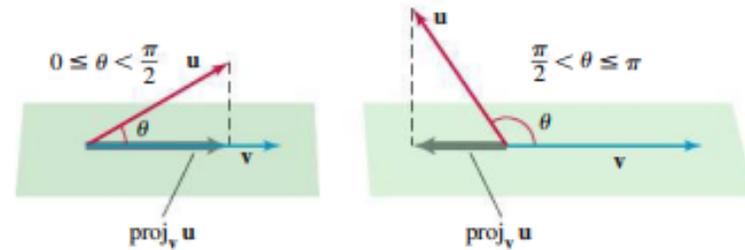
**Extra Example.** Find a unit vector that is orthogonal to both  $\vec{i} + \vec{j}$  and  $\vec{i} + \vec{k}$ .

### Scalar and vector projections



**Definition.** The scalar projection of  $\vec{u}$  onto  $\vec{v}$  is given by

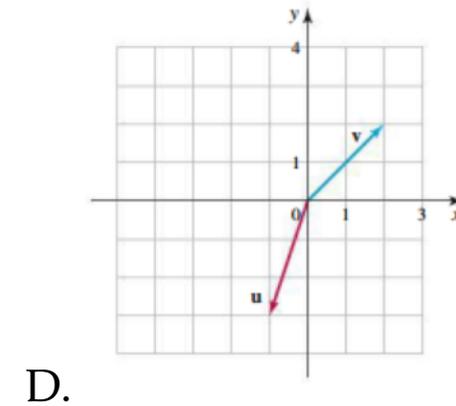
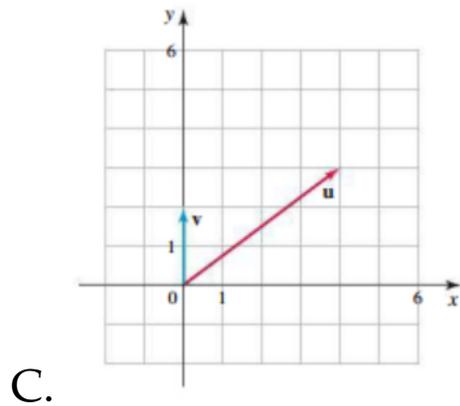
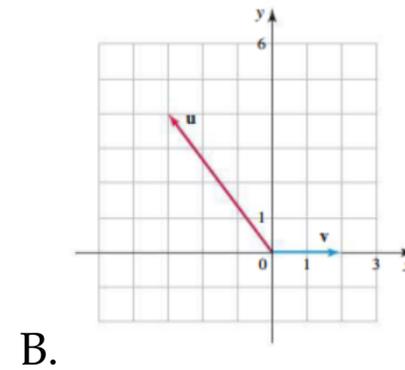
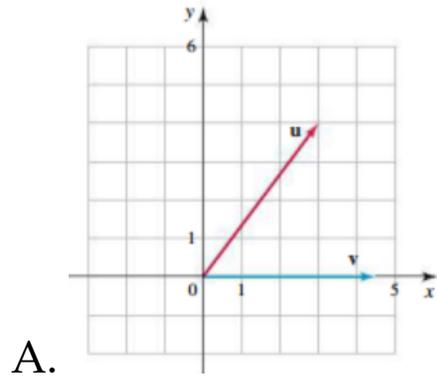
$$scal_{\vec{v}}\vec{u} =$$



**Definition.** The vector projection of  $\vec{u}$  onto  $\vec{v}$  is given by

$$proj_{\vec{v}}\vec{u} =$$

**Example.** Find the scalar and vector projections of  $\vec{u}$  onto  $\vec{v}$  by inspection, without computing any formulas.

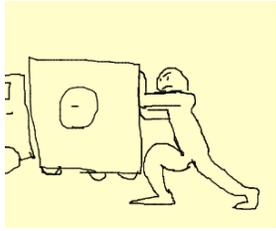


**Example.** Find the scalar and vector projections of  $\vec{b}$  onto  $\vec{a}$ , where

$$\vec{b} = \langle -2, 3, -6 \rangle$$

$$\vec{a} = \langle 5, -1, 4 \rangle$$

**Example.** For the vectors  $\vec{u} = \langle -4, -1, -1 \rangle$  and  $\vec{v} = \langle 2, -2, -2 \rangle$ , express  $\vec{u}$  as the sum  $\vec{u} = \vec{p} + \vec{n}$ , where  $\vec{p}$  is parallel to  $\vec{v}$  and  $\vec{n}$  is orthogonal to  $\vec{v}$ .



**Recall:** The work done by a constant force, moving an object in the direction of the force, is:

**Definition.** The **work** done by a constant force  $\vec{F}$  moving an object along a vector  $\vec{D}$  (not necessarily in the direction of the force) is:

**Question.** How can we write work in terms of the dot product?

**Example.** A tow truck drags a stalled car along a road. The chain makes an angle of  $30^\circ$  with the road and the tension in the chain is 500N. How much work is done by the truck in pulling the car 3 m?

**Extra Example.** Prove that for two lines that are not horizontal or vertical, the two lines are perpendicular if and only if their slopes are ...

**Extra Example.** True or False and justify your answer.

$$|\vec{a} \circ \vec{b}| \leq \|\vec{a}\| \cdot \|\vec{b}\|$$

## §13.4 Cross Products

After completing this section, students will be able to:

- Compute the cross-product of two 3-dimensional vectors from their components.
- Compute the magnitude of the cross-product from the magnitude of the vectors and the angle between them.
- Use the right hand rule to find the direction of the cross-product.
- Use cross product to find a vector perpendicular to two other vectors.
- Use cross product to determine if two vectors are parallel.
- Use cross product to find the area of a parallelogram or triangle.
- Use properties of cross product to determine if statements about vectors are true or false.

**Definition.** The cross-product of two vectors  $\vec{a} = \langle a_1, a_2, a_3 \rangle$  and  $\vec{b} = \langle b_1, b_2, b_3 \rangle$  is given by:

$$\vec{a} \times \vec{b} =$$

**Example.** For  $\vec{a} = \langle 1, 2, 3 \rangle$  and  $\vec{b} = \langle 5, -1, 10 \rangle$ , find  $\vec{a} \times \vec{b}$ .

**Properties of Cross Product:**

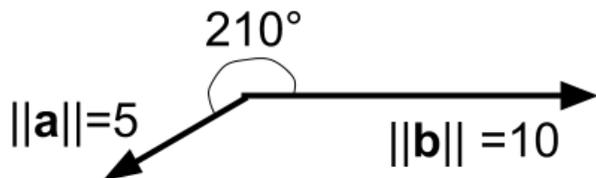
- If  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$ , with  $0 \leq \theta \leq \pi$ , then the length of  $\vec{a} \times \vec{b}$  is given by

$$\|\vec{a} \times \vec{b}\| =$$

- The vector  $\vec{a} \times \vec{b}$  is perpendicular to ...

- The direction of  $\vec{a} \times \vec{b}$  is given by ...

**Example.** For the two vectors shown, find  $\|\vec{a} \times \vec{b}\|$  and determine whether  $\vec{a} \times \vec{b}$  is directed into the page or out of the page.



**Proposition.** If  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$ , with  $0 \leq \theta \leq \pi$ , then

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta$$

**Proposition.** The vector  $\vec{a} \times \vec{b}$  is perpendicular to both  $\vec{a}$  and  $\vec{b}$ .

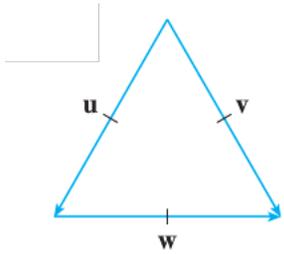
**Proposition.** The direction of  $\vec{a} \times \vec{b}$  is given by the right hand rule.

END OF VIDEOS

**Review.** The cross-product of two vectors  $\vec{a} = \langle a_1, a_2, a_3 \rangle$  and  $\vec{b} = \langle b_1, b_2, b_3 \rangle$  is defined in terms of components as:

or in terms of length and direction as:

**Example.** Find the cross product  $\vec{u} \times \vec{w}$  where  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  are vectors of length 5 that lie in the x-y plane.



**True or False:** If two nonzero vectors  $\vec{a}$  and  $\vec{b}$  are parallel, then  $\vec{a} \times \vec{b} = \vec{0}$ .

**True or False:** If two nonzero vectors  $\vec{a}$  and  $\vec{b}$  have cross-product  $\vec{a} \times \vec{b} = \vec{0}$ , then  $\vec{a}$  and  $\vec{b}$  are parallel.

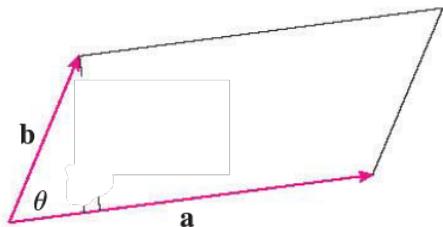
**Question.** How can you use cross product to:

- find a vector perpendicular to two vectors?
  
- determine if two vectors are parallel?

**Question.** Is it possible to find the exact angle between two vectors if all we know is the cross product and the magnitudes of the vectors?

**Example.** Find a unit vector orthogonal to both  $\vec{i} + \vec{j}$  and  $\vec{i} + \vec{k}$ .

**Example.** Use the cross product to write the area of the parallelogram in terms of  $\vec{a}$  and  $\vec{b}$ .



**Example.** Find the area of the triangle with vertices  $P(0, 0, -3)$ ,  $Q(4, 2, 0)$ , and  $R(3, 3, 1)$ .

**True or False**

1.  $\vec{a} \times \vec{b}$  is a scalar.

2.  $\vec{a} \times \vec{a} = \vec{0}$ .

3. For  $\vec{a}, \vec{b} \neq \vec{0}$ , if  $\vec{a} \times \vec{b} = \vec{0}$  then  $\vec{a} = \vec{b}$ .

4.  $\vec{i} \times \vec{j} = \vec{k}$

5.  $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$

6.  $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$

## Properties of Cross Product:

Let  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  be nonzero vectors in  $\mathbb{R}^3$ , and let  $a$  and  $b$  be scalars.

1.  $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$  Anticommutative property
2.  $(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v})$  Associative property
3.  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$  Distributive property
4.  $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$  Distributive property

**Extra Example.** Are these three vectors coplanar?

$$\vec{u} = 2\vec{i} + 3\vec{j} + \vec{k}$$

$$\vec{v} = \vec{i} - \vec{j}$$

$$\vec{w} = 7\vec{i} + 3\vec{j} + 2\vec{k}$$

**Extra Example.** 1. Find all vectors  $\vec{v}$  such that  $\langle 1, 2, 1 \rangle \times \vec{v} = \langle 3, 1, -5 \rangle$ .

2. Explain why there is no vector  $\vec{v}$  such that  $\langle 1, 2, 1 \rangle \times \vec{v} = \langle 3, 1, 5 \rangle$ .

Summary of geometric and physical applications of dot-product and cross-product:

## §13.5 Lines and Planes

After completing this section, students should be able to:

- Recognize equations of lines in parametric form, symmetric form, and vector form.
- Recognize equations of planes.
- Write the equation of a line given its direction and a point on the line, or given two points, or similar information.
- Write the equation of a plane given three points, or one point and a normal vector, or through a line and a point, or through two intersecting or parallel lines, or from similar information.
- Determine if planes intersect and find lines of intersection and angles of intersection.
- Determine if lines intersect, are parallel, or are skew.
- Use scalar projection to find the distance from a point to a plane, or a point to a line, or between two skew or parallel lines, or between two planes, or similar configurations.

**Example.** Is the line through  $(-4, -6, 1)$  and  $(-2, 0, -3)$  parallel to the line through  $(10, 18, 4)$  and  $(5, 3, 14)$ ?

What is the equation of the line through the origin, that is parallel to the line through  $(-4, -6, 1)$  and  $(-2, 0, -3)$ ?

What is the equation of the line through  $(-4, -6, 1)$  and  $(-2, 0, -3)$ ?

How else could you write an equation of the line through  $(-4, -6, 1)$  and  $(-2, 0, -3)$ ?

**Note.** The equation of a line through the point  $(x_0, y_0, z_0)$  in the direction of the vector  $\langle a, b, c \rangle$  can be described:

with the parametric equations:

or, with "symmetric equations":

or, with the vector equation:

**Example.** Find the equation of the plane through the point  $(-3, 2, 0)$  and perpendicular to the vector  $\langle 1, -2, 5 \rangle$

**Note.** The plane through the point  $(x_0, y_0, z_0)$  and perpendicular to the vector  $\langle a, b, c \rangle$  is given by the equation

END OF VIDEO

**Review.** Which of these equations represents a line in 3-dimensional space?

1.  $3x + 5y = 2$

2.  $3(x - 1) + 5(y - 3) - 4(z - 2) = 0$

3.  $x = 7 + 4t, y = 5 - 3t, z = 7t$

4.  $\frac{x-7}{4} = \frac{-y+5}{3} = \frac{z}{7}$

5.  $\vec{r}(t) = \langle 7 + 4t, 5 - 3t, 7t \rangle$

**Review.** The equation of a line in 3-dimensional space can be described with parametric equations:

with "symmetric equations":

with the vector equation:

where  $\langle a, b, c \rangle$  represents ...

and  $(x_0, y_0, z_0)$  represents ...

**Review.** The equation of a plane in 3-dimensional space can be described by:

where  $\langle a, b, c \rangle$  represents ...

and  $(x_0, y_0, z_0)$  represents ...

**Example.** Find an equation of the line through the point  $(3, -1, 2)$  and perpendicular to the plane  $4x - 6y + z = 13$ .

**Example.** Find an equation for the plane through the points  $(3, -1, 2)$ ,  $(8, 2, 4)$ , and  $(-1, -2, -3)$ .

**Question.** What information is enough to determine the equation for a line *in the plane*?

**Question.** What information is enough to determine the equation for a line *in 3-dimensional space*?

**Question.** What information is enough to determine the equation for a plane in 3-dimensional space?

**Extra Example.** Find the equation of the plane that is perpendicular to the two planes  $x - 2y + 3z = 4$  and  $x + y = 7$  and goes through the point  $(5, -3, 1)$ .

**Example.** Find the line of intersection of the planes  $2x - y + z = 5$  and  $x + y + z = 1$ .

**Example.** Determine whether the lines  $L_1$  and  $L_2$  are parallel, skew, or intersecting.

$$L_1 : x = 3 + 2t, y = 4 - t, z = 1 + 3t$$

$$L_2 : x = 1 + 4s, y = 3 - 2s, z = 4 + 5s$$

**Example.** Determine whether the planes are parallel, perpendicular or neither. If neither, find the angle between them.

$$x + 2y + 2z = 1, \quad 2x - y + 2z = 1$$

**Example.** Find the distance between the point  $(1, 2, 3)$  and the plane  $5x + 4y - 3z = 10$ .

**Extra Example.** Find the distance between the skew lines

$$L_1 : x = 3 + 2t, y = 4 - t, z = 1 + 3t$$

$$L_2 : x = 1 + 4s, y = 3 - 2s, z = 4 + 5s$$

**Example.** Which of these equations represents the same line in 3-d space as

$$\frac{x-3}{5} = \frac{y+2}{7} = \frac{z}{2}$$

A.  $x = 5t + 3, y = 7t - 2, z = 2t$

B.  $x = 10t + 3, y = 14t - 2, z = 4t$

C.  $x = 5t + 8, y = 7t + 5, z = 2t + 2$

D.  $x = 5t^3 + 3, y = 7t^3 - 2, z = 2t^3$

E.  $x = 5t - 2, y = 7t + 4, z = 2t + 1$