§15.1 Functions of Two or More Variables

After completing this section, students should be able to:

- Match equations of the form $z = f(x, y)$ to graphs of surfaces and graphs of level curves.
- Describe the graphs of functions of three variables $w = f(x, y, z)$ in terms of the level curves $f(x, y, z) = k$

Example. Consider the function of two variables $f(x, y) = \sqrt{xy}$

1. What is its domain?

For the function $f(x, y) = \sqrt{xy}$ ...

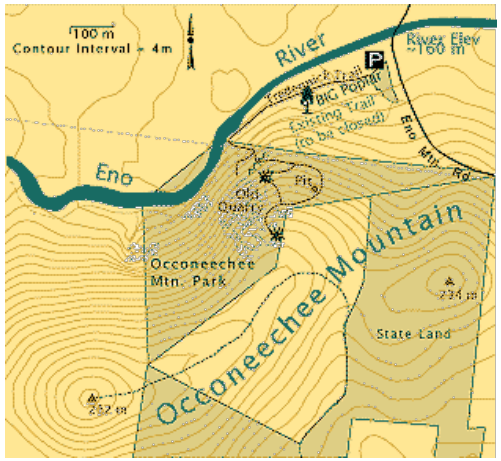
2. What are its level curves?

For the function $f(x, y) = \sqrt{xy}$ ...

3. What does its graph look like?

END OF VIDEO

Question. What do the lines in this contour map represent? Where should you go if you like steep hills? Mountain tops?

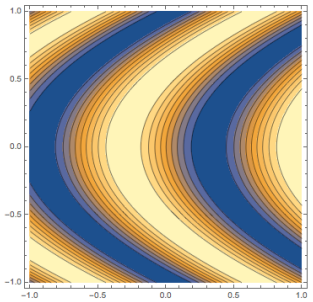
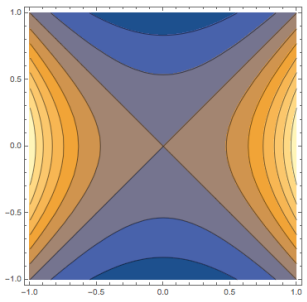
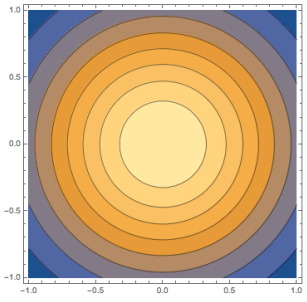
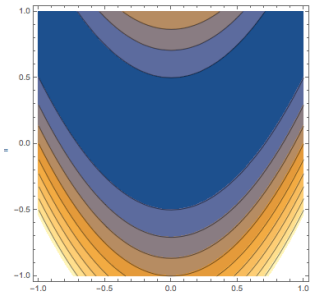


Example. What type of curve do you get when you intersect the graph of $z = (x^2 - y)^2$ with a horizontal plane?

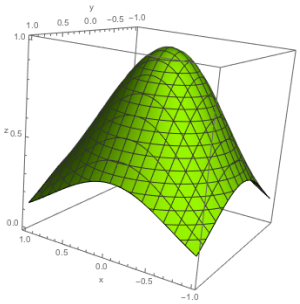
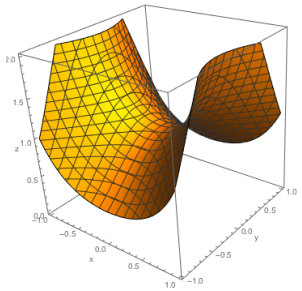
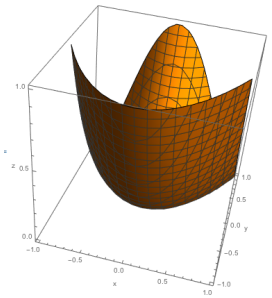
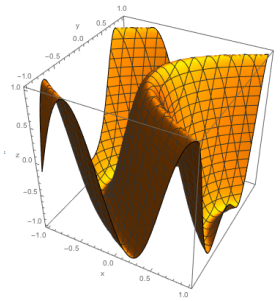
Definition. A **level curve** of a surface $z = f(x, y)$ is ...

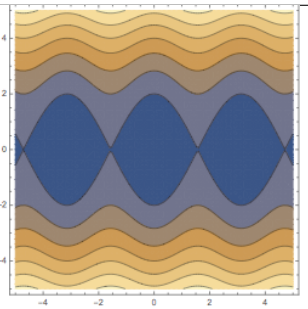
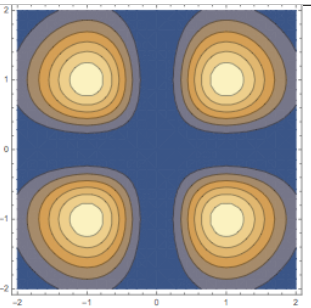
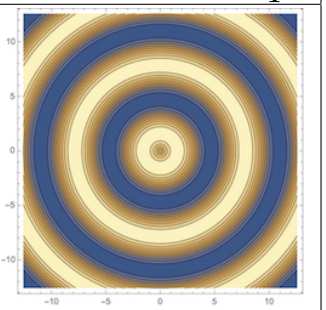
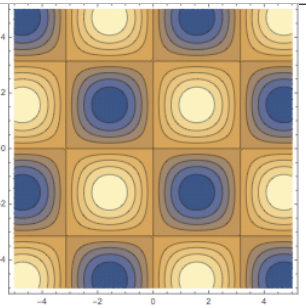
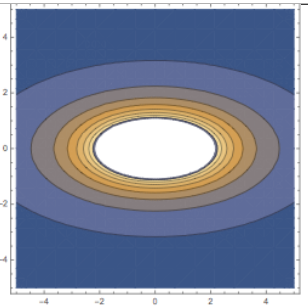
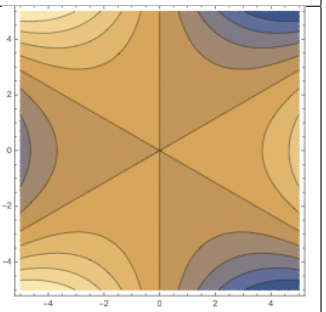
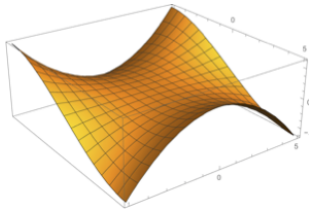
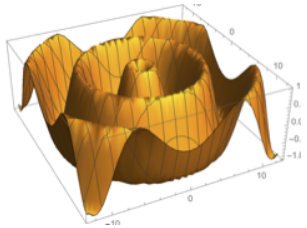
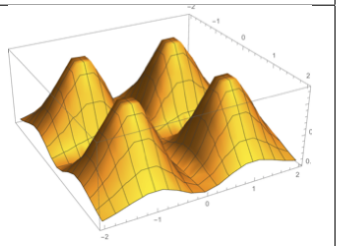
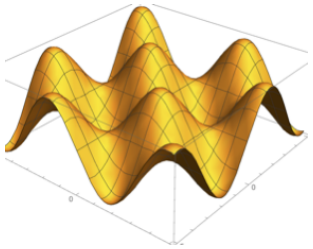
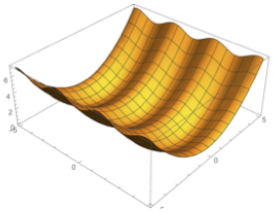
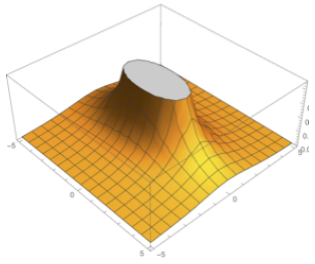
Definition. A **contour map** of a surface $z = f(x, y)$ is ...

Example. Find the contour map for the surface $z = (x^2 - y)^2$.

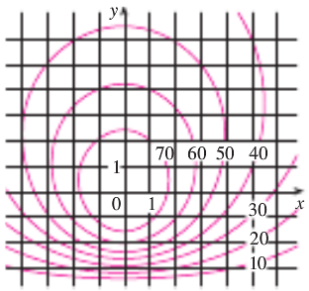
A.  B.  C.  D. 

Example. Find the 3-d graph of the surface $z = (x^2 - y)^2$.

1.  2.  3.  4. 

1. $z = \sin\sqrt{x^2+y^2}$	2. $z = \frac{1}{x^2+4y^2}$	3. $z = \sin(x)\sin(y)$
4. $z = x^2y^2e^{-(x^2+y^2)}$	5. $z = x^3 - 3xy^2$	6. $z = \sin^2(x) + \frac{y^2}{4}$
I. 	II. 	III. 
IV. 	V. 	VI. 
A. 	B. 	C. 
D. 	E. 	F. 

Example. A contour map for a function f is shown. Use it to estimate the values of $f(-3,3)$ and $f(3,-2)$. What can you say about the shape of the graph?



Functions of 3 or more variables

To visualize functions $f(x, y, z)$ of three variables, it is handy to look at **level surfaces**.
Example. $f(x, y, z) = x^2 + y^2 + z^2$

(a) Guess what the level surfaces should look like.

(b) Graph a few level surfaces (e.g. $x^2 + y^2 + z^2 = 10$, $x^2 - y^2 + z^2 = 20$, $x^2 - y^2 + z^2 = 30$) on a 3-d plot.

Example. $f(x, y, z) = x^2 - y^2 + z^2$

1. Guess what the level surfaces should look like.
2. Graph a few level surfaces (e.g. $x^2 - y^2 + z^2 = 0$, $x^2 - y^2 + z^2 = 10$, $x^2 - y^2 + z^2 = 20$) on a 3-d plot.

§15.2 Limits and Continuity

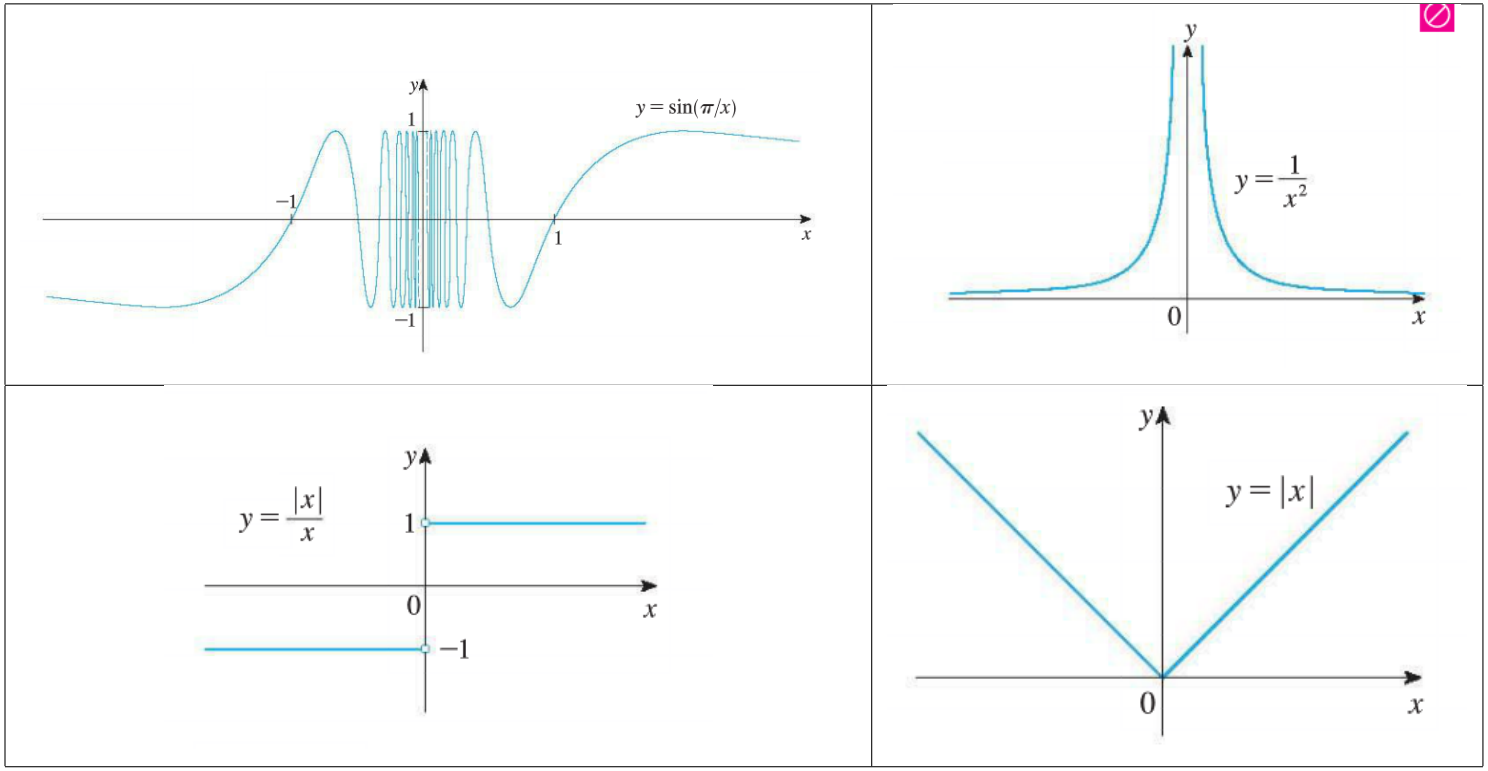
After completing this section, students should be able to:

1. Use a graph to build intuition for whether or not a limit of a function of two variables exists.
2. Prove that a limit of a function of two variables does not exist by finding two paths along which the function approaches different values.
3. Prove that a limit of a function of two variables does exist by converting to polar coordinates and using the squeeze theorem.
4. State the definition of continuity for functions of two variables in terms of limits.
5. Determine where a function is continuous.

Recall LIMITS from Calculus 1:

Informally, $\lim_{x \to a} f(x) = L$ if the y-values $f(x)$ get closer and closer to the same number L when x approaches a from either the left of the right.

Does $\lim_{x \to 0} f(x)$ exist for these functions?

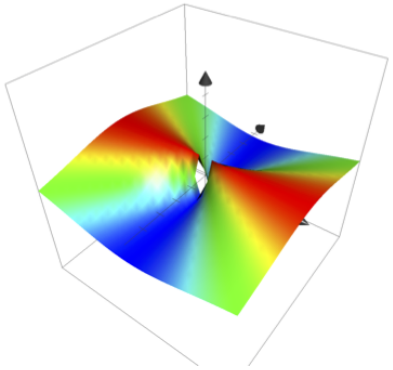


LIMITS for functions of two variables:

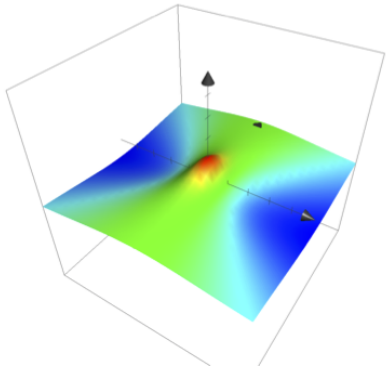
Informally, $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if the z-values $f(x,y)$ ...

For each function, decide if $\lim_{(x,y)\to(0,0)} f(x,y)$ exists. The color is based on height: low is blue and high is red.

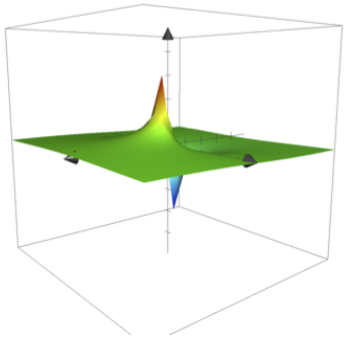
A) $\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$



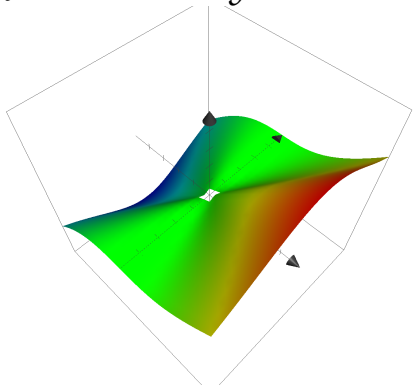
B) $\lim_{(x,y)\to(0,0)} \dfrac{1-x^2}{x^2+y^2+1}$



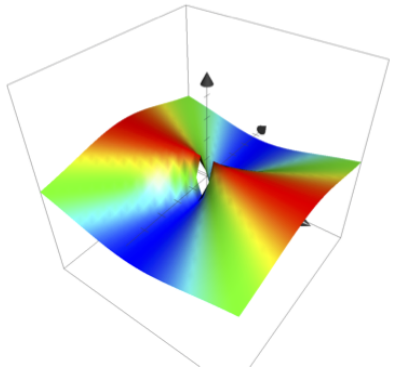
C) $\lim_{(x,y)\to(0,0)} \dfrac{x}{x^2+y^2}$



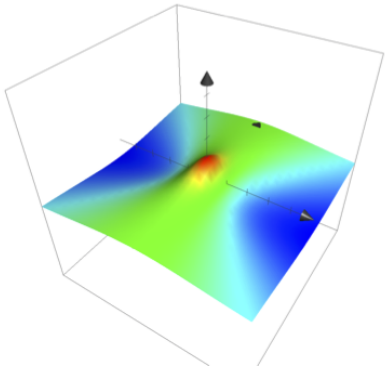
D) $\lim_{(x,y)\to(0,0)} \dfrac{x^3}{x^2+y^2}$



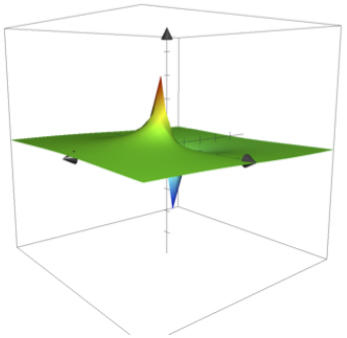
Example. $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$



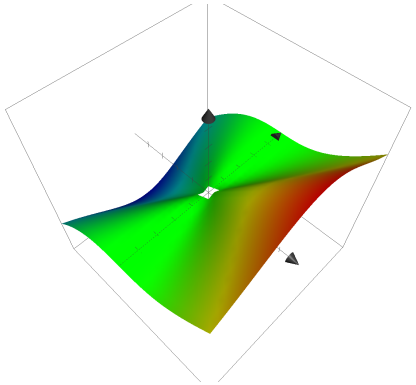
Example. $\lim_{(x,y)\to(0,0)} \dfrac{1-x^2}{x^2+y^2+1}$



Example. $\lim_{(x,y)\to(0,0)} \dfrac{x}{x^2+y^2}$

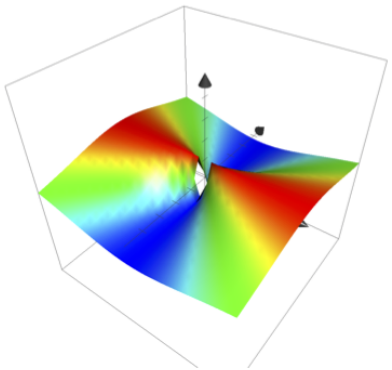


Example. $\lim_{(x,y)\to(0,0)} \dfrac{x^3}{x^2+y^2}$

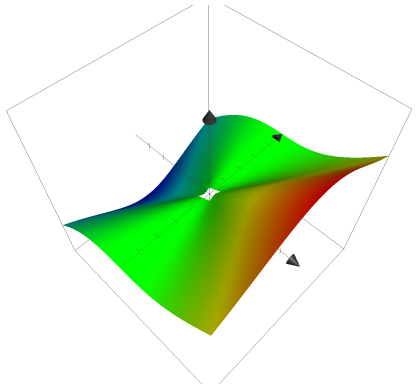


END OF VIDEO

Review. Informally, $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if the z-values $f(x,y)$ ...



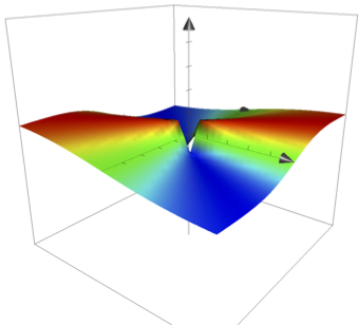
$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$



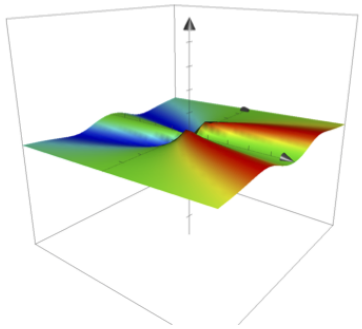
$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2 + y^2}$$

Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? The color is based on height: low is blue and high is red.

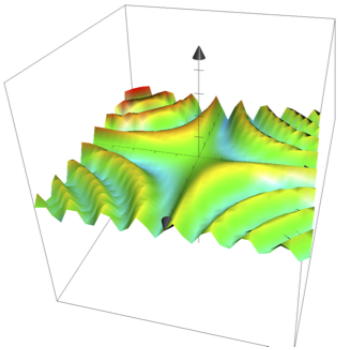
A) $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2}$



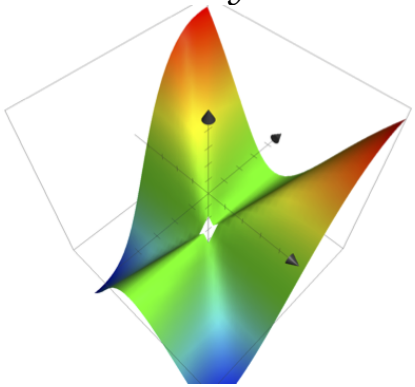
B) $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$



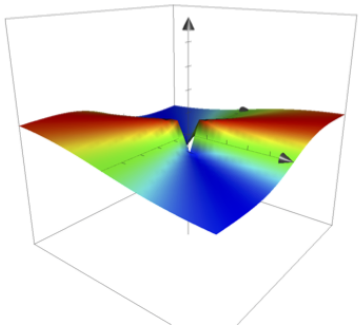
C) $\lim_{(x,y)\to(0,0)} \dfrac{10\sin(xy)}{x+y+20}$



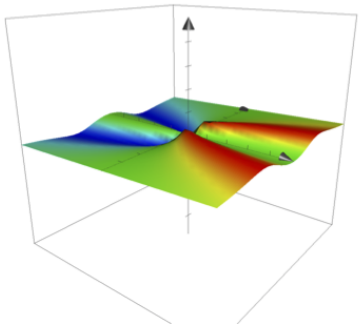
D) $\lim_{(x,y)\to(0,0)} \dfrac{3x^2y-3x^2-3y^2}{x^2+y^2}$



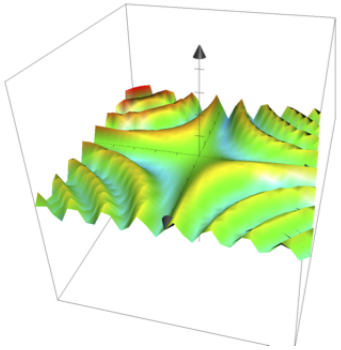
Example. Show that $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2}$ does not exist.



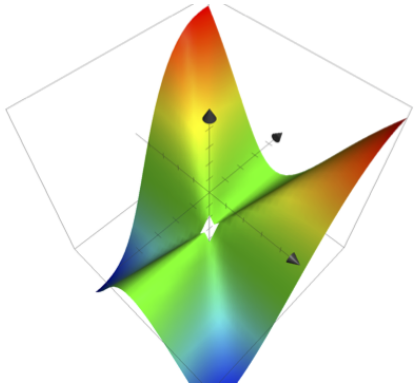
Example. Show that $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ does not exist.



Example. Show that $\lim_{(x,y)\to(0,0)} \dfrac{10\sin(xy)}{x+y+20}$ does exist.



Example. Show that $\lim_{(x,y)\to(0,0)} \dfrac{3x^2y - 3x^2 - 3y^2}{x^2 + y^2}$ does exist.



Summary: for practical purposes, the best way to show that a limit does not exist is to:

For practical purposes, the best ways to show that a limit exists is to:

-
-
-
-

Recall CONTINUITY from Calculus 1: A function f is continuous at the point $x = a$ if:

1.
2.
3.

Recall from Calculus 1: common functions that are continuous include:

CONTINUITY for functions of of two (or more) variables:

A function $f(x, y)$ is continuous at the point $(x, y) = (a, b)$ if:

1.
2.
3.

Common functions that are continuous include:

-
-
-
-

Example. Where is $f(x, y) = \dfrac{\sqrt{4 - x^2} + \sqrt{4 - y^2}}{1 - x^2 - y^2}$ continuous?

§15.3 Partial Derivatives

After completing this section, students should be able to:

- Compute partial derivatives.
- Use average rates of change to approximate partial derivatives.
- For functions of two variables, explain the geometric meaning of a partial derivative as the slope of a tangent line to a curve in the intersection of a surface and a plane.

Example. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. So we write $h = f(v, t)$.

Duration (hours)

Wind speed (knots) v \ t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

1. What is $f(40, 20)$?

2. If we fix duration at $t = 20$ hours and think of $g(v) = f(v, 20)$ as a function of v, what is the approximate value of the derivative $\left.\frac{dg}{dv}\right|_{v=40}$?

3. If we fix wind speed at 40 knots, and think of $k(t) = f(40, t)$ as a function of duration t, what is the approximate value of the derivative $\left.\frac{dk}{dt}\right|_{t=20}$?

Definition. For a function $f(x, y)$ defined near (a, b), the **partial derivatives** of f at (a, b) are:

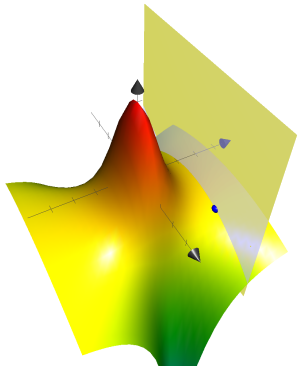
$f_x(a, b)$ = the derivative of $f(x, b)$ with respect to x when $x = a$, and

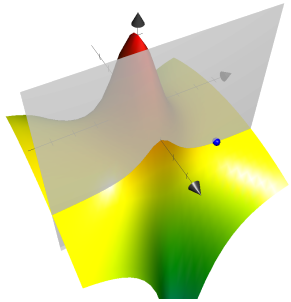
$f_y(a, b)$ = the derivative of $f(a, y)$ with respect to y when $y = b$.

In terms of the limit definition of derivatives, we have:

$$f_x(a, b) = \lim_{h \to 0}$$
$$f_y(a, b) = \lim_{h \to 0}$$

Geometrically, $f(x, b)$ can be thought of as





So $f_x(a, b) = \frac{d}{dx} f(x, b)|_{x=a}$ can be thought of as

Note. To compute f_x, we just take the derivative with x as our variable, holding all other variables constant. Similarly for the partial derivative with respect to any other variable.

Example. $f(x, y) = \frac{x}{y}$. Find $f_x(1, 2)$ and $f_y(1, 2)$.

Notation. There are many notations for partial derivatives, including the following:

f_x	$\frac{\partial f}{\partial x}$	$\frac{\partial z}{\partial x}$	f_1	$D_1 f$	$D_x f$

Note. Partial derivatives can also be taken for functions of three or more variables. For example, if $f(x, y, z, w)$ is a function of 4 variables, then $f_z(3, 4, 2, 7)$ means:

END OF VIDEO

Review. Which of the following is the limit definition of $\dfrac{\partial f}{\partial x}(1,2)$?

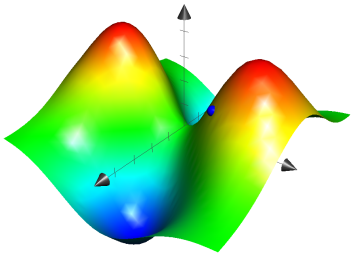
A. $\displaystyle\lim_{h\to 0} \frac{f(1+h,2+h)-f(1,2)}{h}$

B. $\displaystyle\lim_{h\to 0} \frac{f(1+h,2)-f(1,2+h)}{h}$

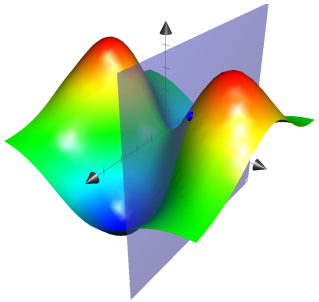
C. $\displaystyle\lim_{h\to 0} \frac{f(1+h,2)-f(1,2)}{h}$

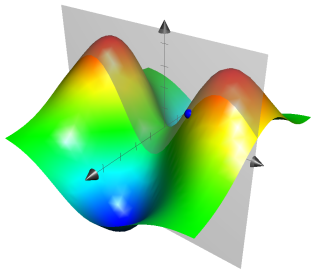
D. $\displaystyle\lim_{h\to 0} \frac{f(1,2+h)-f(1,2)}{h}$

Review. Based on the graph of $z = f(x, y)$ shown



- is $\dfrac{\partial f}{\partial x}(1, 2)$ positive, negative, or 0?
- is $\dfrac{\partial f}{\partial y}(1, 2)$ positive, negative, or 0?





Question. For $f(x, y) = 3x^2y + 5y^2 + \cos(xy)$, find $\dfrac{\partial f}{\partial x}$.

Example. For $f(x, y, z) = e^{x \sin(y-z)}$, find $D_x f$ and $D_z f$.

Example. For $z = f(x)g(y)$, find an expression for $\dfrac{\partial z}{\partial x}$.

Example. For $x^2 - y^2 + z^2 - 2z = 12$, find $\dfrac{\partial z}{\partial y}$.

Notation. Second derivatives are written using any of the following notations:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Example. For $f(x, y) = x^2 + x^2y^2 - 2y^2$, calculate f_{xx}, f_{xy}, f_{yx}, and f_{yy}.

Theorem. (Clairout's Thm) Let $f(x, y)$ be defined on a disk D containing (a, b). If f_{xy} and f_{yx} are both defined and _____ on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

Example. Consider the function

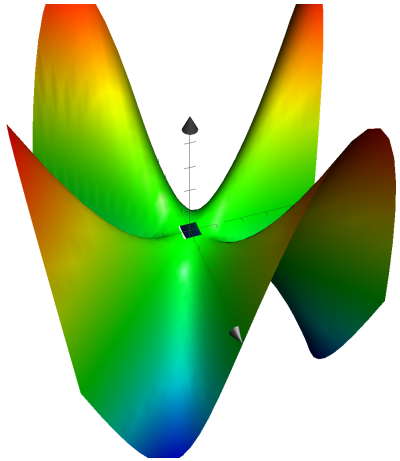
$$f(x, y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

It turns out that

- for all x $D_2 f(x, 0) =$
- for all y $D_1 f(0, y) =$

Therefore,

- $D_{2,1} f(0, 0) =$
- $D_{1,2} f(0, 0) =$



Question. True or False: There is a function f with continuous second partial derivatives such that $f_x = x$ and $f_y = x$.

Extra Example. The wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

where $u(x, t)$ represents displacement, t represents time, x represents the distance from one end of the wave, and a is a constant.

Verify that the function $u(x, t) = \sin(x - at)$ satisfies the wave equation.

In Calc 1, a function is differentiable means that $f'(x)$ exists.

Intuitively, when a function is is differentiable, it has a tangent line that is a good approximation of the function.

Question. True or False (make a guess): A function $f(x, y)$ of two variables is differentiable means that f_x and f_y exists.

Question. True or False (make a guess): Intuitively, if a function of two variables is differentiable, then it has a tangent plance that is a good approximation of the function.

§15.4 The Chain Rule

After completing the section, students should be able to:

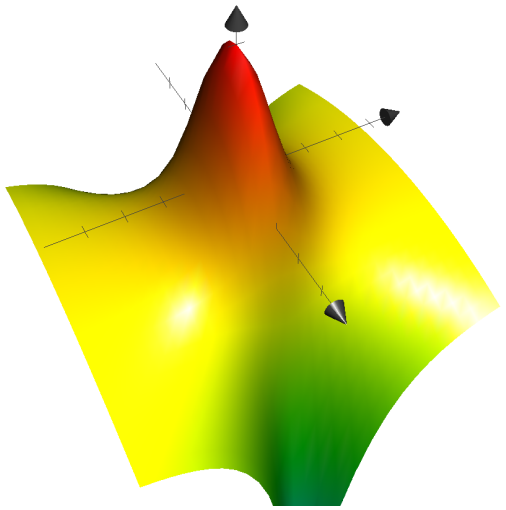
- Compute partial derivatives using the Chain Rule, using formulas for functions, or using numerical information such as tables of values or contour graphs.

Remember the Chain Rule for functions of 1 variable:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Theorem. (Chain Rule, Case 1) Suppose $z = f(x, y)$ is a differentiable function of x and y and $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then

$$\frac{dz}{dt} =$$



Example. $w = x\cos(4y^2z), x = t^2, y = 1 - t, z = 1 + 2t$. Find $\frac{dw}{dt}$.

Theorem. (Chain Rule, Case 2) Suppose $z = f(x, y)$ is a differentiable function of x and y and $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of t. Then

$$\frac{\partial z}{\partial s} =$$

$$\frac{\partial z}{\partial t} =$$

Example. $z = e^r \cos(\theta), r = st, \theta = \sqrt{s^2 + t^2}$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

END OF VIDEO

Review. The Chain Rule If u is a differentiable function of variables x, y, z, and each of x, y, and z are differentable functions of variables r, t, then

$$\frac{\partial u}{\partial r} =$$

$$\frac{\partial u}{\partial t} =$$

Example. $w = \ln \sqrt{x^2 + y^2 + z^2}$, $x = \sin t$, $y = \cos t$, $z = \tan t$. Find $\dfrac{dw}{dt}$.

Example. $u = xe^{ty}$, $x = \alpha^2\beta$, $y = \beta^2\gamma$, $t = \gamma^2\alpha$. Find $\dfrac{\partial u}{\partial \alpha}$ when $\alpha = -1$, $\beta = 2$, $\gamma = 1$.

Example. A manufacturer modeled its yearly production P in millions of dollars as a Cobb-Douglas function

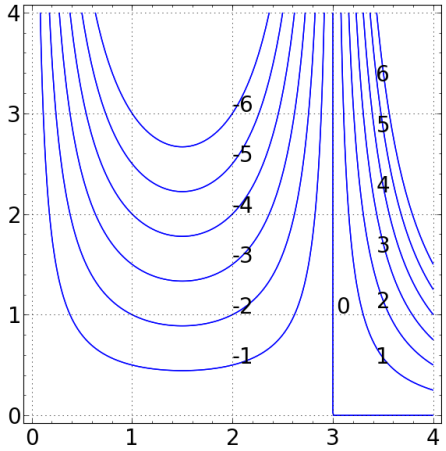
$$P(L, K) = 1.47L^{0.65}K^{0.35}$$

where L is the number of labor hours (in thousands) and K is the invested capital (in millions of dollars). Suppose that when L is 30 and K is 8, the labor force is decreasing at a rate of 2000 labor hours per year and capital is increasing at a rate of \$500,000 per year. Find the rate of change of production per year. Note the units.

Extra Example. Suppose $f(x, y) = h(x^2 - y^2, 3xy^2 - 4)$, where $h(u, v)$ is a differentiable function.

1. Find equations for f_x and f_y in terms of h_u and h_v.

2. Use the contour graph for h to estimate f_x and f_y at the point $(x, y) = (2, 1)$.



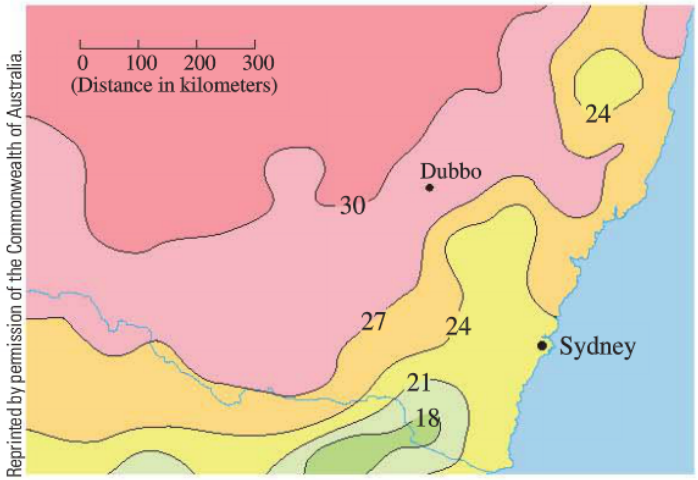
This is a contour graph for h not f.

§15.5 Directional Derivatives and the Gradient

After completing this section, students should be able to ...

- Find the directional derivative of a function of two or more variables, in the direction of a vector.
- Find the gradient of a function of two or more variables.
- Explain how to calculate the directional derivative from the gradient.
- Find the direction of greatest increase for a function and the magnitude of that increase.
- Describe the relationship between the gradient vector and the level curves of a function.
- Write down the equation for the tangent line for a level curve of a function of two variables, and the equation of the tangent plane to the level surface of a function of three variables, at a specified point.
- Write down the equation for a normal line to a level curve or level surface at a specified point.
- Find the path of steepest descent on a surface.

Example. What is the approximate rate of change of the temperature function $f(x, y)$ at Dubbo, New South Wales, in the direction of Sydney?



Definition. The directional derivative of f at (x_0, y_0) in the direction of the unit vector $\vec{u} =< a, b >$ is given by:

$$D_{\vec{u}} f(x_0, y_0) =$$

if this limit exists.

Equivalently,

Definition. The directional derivative of f at (x_0, y_0) in the direction of the unit vector $\vec{u} =< a, b >$ is given by:

Assuming f is differentiable, what does the Chain Rule tell us about this derivative?

Theorem. For the unit vector $\vec{u} =< a, b >$,

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0) _____ + f_y(x_0, y_0) _____$$

Note. For a vector $\vec{v} =< c, d >$ that is NOT a unit vector, the directional derivative in the direction of $\vec{v}$ is defined as ...

Example. Find the directional derivative of $f(x, y) = x\cos(y)$ at $(2, 0)$ in the direction of $\vec{v} =< 3, 4 >$

END OF VIDEO

Review. For a unit vector $\vec{u} =< a, b >$, the directional derivative $D_{\vec{u}} f(x_9, y_0)$ is given by which of the following? (select all that apply)

A. $\lim_{t \to 0} \dfrac{f(x_0 + at, y_0 + bt) - f(x_0, y_0)}{t}$

B. $\left.\dfrac{dg}{dt}\right|_{t=o}$, where $g(t) = f(x_0 + at, y_0 + bt)$

C. $af_x(x_0, y_0) + bf_y(x_0, y_0)$

D. The slope of the curve of intersection of the surface $z = f(x, y)$ and the vertical plane that contains the vector $< a, b >$, if the curve is traversed in the direction of the vector $< a, b >$.

Question. True or False: It is possible for $D_u(f(x_0, y_0)$ to be positive for all unit vectors $\vec{u}$. (Assume f is differentiable.)

Example. Find the directional derivative of $f(x, y) = \sqrt{2x + 3y}$ when $(x, y) = (3, 1)$

- in the direction $\theta = -\frac{\pi}{6}$

- in the direction of the vector $\vec{w} =< 5, 7 >$

Definition. The **gradient** of the function $f(x, y)$ at (x_0, y_0) is defined as:

$$\nabla f(x_0, y_0) =$$

Note. For a unit vector $\vec{u} =< a, b >$, $D_{\vec{u}} f(x_0, y_0)$ can be written in terms of the gradient as:

$$D_{\vec{u}} f(x_0, y_0) =$$

Example. For $f(x, y) = xe^{-y} - y$,

1. Find ∇f at the point $(2, 0)$.
2. Find the directional derivative at $(2, 0)$, in the direction of $\vec{v} =< 3, -4 >$.

Question. For a differentiable function f, at a point (x_0, y_0), in what direction is f increasing most steeply? (i.e. for what unit vector $\vec{u}$ is $D_{\vec{u}}f$ maximal?) Answer in terms of the gradient.

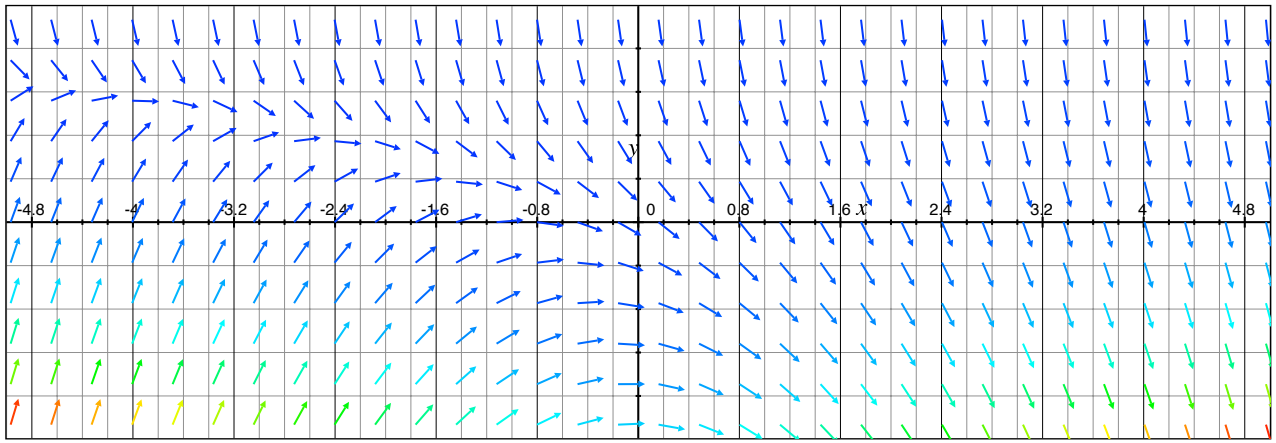
Theorem. The maximum value (most positive value) of $D_{\vec{u}}$ is _____ and this maximum occurs in the direction of _____ .

The minimum value (most negative value) of $D_{\vec{u}}$ is _____ and this minimum occurs in the direction of _____ .

Example. For $f(x, y) = xe^{-y} - y$ at $(2, 0)$, what is the maximum directional derivative and what direction does it occur in?

Note. For $f(x, y) = xe^{-y} - y$, consider the graph of ∇f on the x-y plane drawn below.

Since ∇f points in the direction of greatest increase, where do you expect the level curves to be?



Theorem. For a differentiable function f of two variables, the level curves of f are _____ to ∇f.

Question. Why is ∇f perpendicular to the level curves of f?

Example. Use the gradient to find the equation of the tangent line to the level curve for $f(x, y) = x^2 - y^2$ at $(2, 3)$.

Note. The tangent line for a level curve $f(x, y) = k$ at the point (x_0, y_0) is given by the equation:

Everything we have done so far can also be done for functions of three or more variables! For $f(x, y, z)$, and unit vector $\vec{u} =< a, b, c >$

- $\nabla f(x_0, y_0, z_0) =$

- $D_{\vec{u}} f(x_0, y_0, z_0) =$

- The direction of greatest increase at the point (x_0, y_0, z_0) is _____ , and the largest value of the directional derivative is _____ .

- The direction of greatest decrease at the point (x_0, y_0, z_0) is _____ , and the most negative value of the directional derivative is _____ .

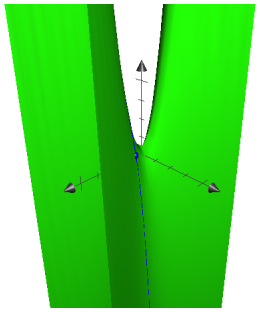
- For any curve in the level surface of f that passes through the point (a, b, c), $\nabla f(a, b, c)$ is perpendicular to ...

- $\nabla f(x, y, z)$ is perpendicular to _____ .

- The equation for a tangent plane to a level surface $f(x, y, z) = k$ at the point (x_0, y_0, z_0) is given by the equation:

Example. Find the equation of the tangent plane to the surface $xy + yz + zx = 5$ at the point $(1, 2, 1)$.

Example. The hyperbolic paraboloid $z = f(x, y) = 1 + x^2 - 3y^2$ is shown below. A ball is released at the point $(1, 2, 18)$ on the surface and it follows the path of steepest descent C.



(a) Find the equation of the projection of C onto the xy-plane.

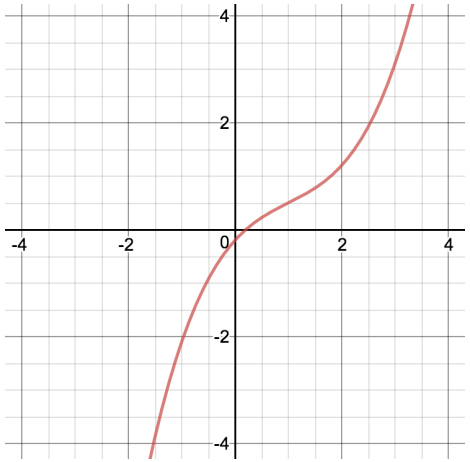
(b) Find the equation of C on the surface.

§15.6 Tangent Planes and Linear Approximations

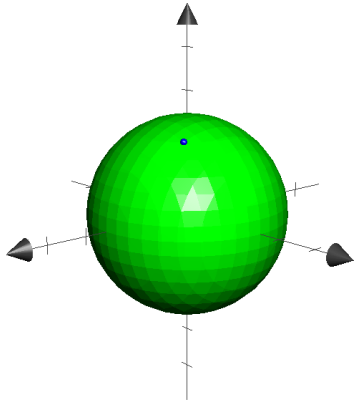
After completing this section, students should be able to:

- Find the equation for the tangent plane of a surface $z = f(x, y)$ at a point.
- Use the tangent plane to approximate a function.
- Compute the differential and use it to estimate errors.
- Explain the relationship between the tangent plane, the linearization equation, and the differential.
- Compute the tangent plane from information about the function's value and partial derivative values.
- Use linearization to interpolate between numerical values in a 2 x 2 talbe.

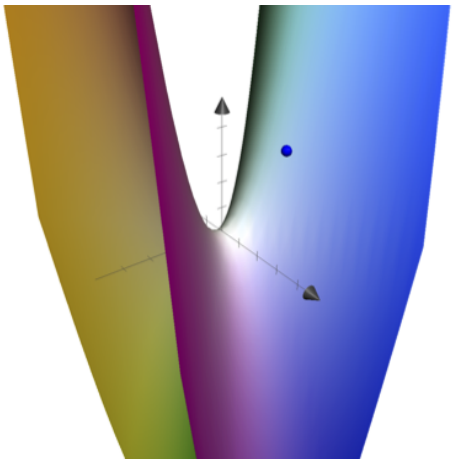
Tangent line:



Tangent plane:



Example. Find the tangent plane to the surface $z = y^2 - x^2$ at the point $(1, 2, 3)$.



Find a general formula for the tangent plane of $z = f(x, y)$ at the point (x_0, y_0, z_0).

Notice that this formula is analogous to the formula for a tangent line for a function of one variable.

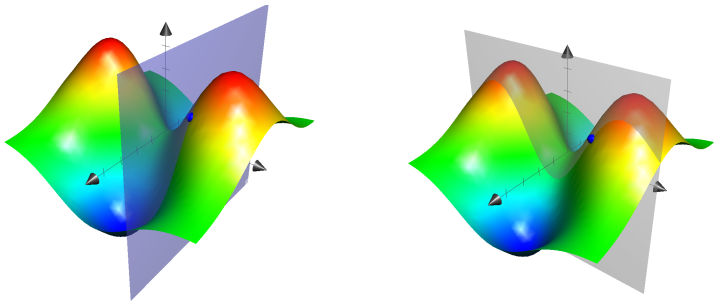
Usually, the tangent plane at a point (x_0, y_0, z_0) is a good approximation of the surface near that point.

Example. Find the tangent plane for the surface $z = f(x, y)$ when $(x, y) = (1, 1)$, where $f(x, y) = 1 - xy\cos(\pi y)$. Use it to approximate $f(1.02, 0.97)$.

END OF VIDEO

NEED TO CHANGE THIS UP – start with review of Calc 1 formulas like in 2019 Fall filed in notes, change up rest so it is not so repetitive, make the four numbers problem a recitation problem? and consider adding one about the change in a quantity given the change in inputs, making one change pos and one neg

Example. Suppose we have a surface $z = f(x, y)$ and we know that $f(1, 2) = 6$, $f_x(1, 2) = -7$ and $f_y(1, 2) = 4$. What is the equation for the tangent plane to the surface at the point $(1, 2, 6)$?



Example. Find the tangent plane for the surface $z = \sin(x) + y^2$ at $(x, y) = (0, 2)$ and use it to approximate $\sin(0.2) + (1.9)^2$

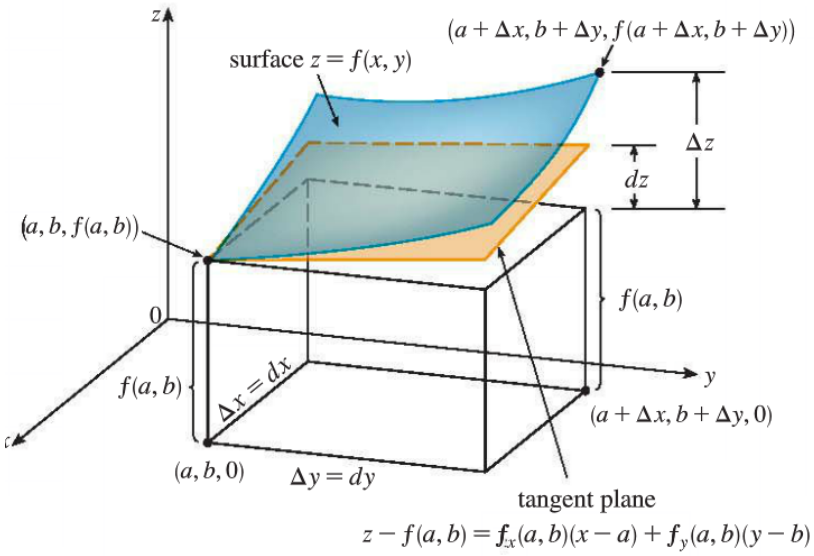
This method of approximating a function's value with the height of the tangent plane can be written in terms of a linearization equation or in the language of differentials.

Definition. The linearization of $f(x, y)$ is written as:

Recall: In Calc 1, for a function $y = f(x)$, the differential was defined as $df = f'(x)dx$.

Definition. The **differential** for the function $y = f(x, y)$ is:

Δf represents the actual change in a function $f(x, y) - f(a, b)$. df represents the corresponding change in the *tangent plane* between (a, b) and (x, y).



Differentials are useful for estimating errors and making approximations, because the differential is linear in all its variables, and linear functions are easier to calculate.

Example. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

Example. Four positive numbers, each less than 50, are rounded to the first decimal place and multiplied together.

1. Use differentials to estimate the size of the maximum possible error in the computed product that might result from the rounding.
2. If the four numbers round to 20, 10, 30, and 5, what is the estimated maximum relative error? percent error?

When the tangent plane approximation works and when it doesn't.

Note. Anytime f_x and f_y exist at a point (a, b), it is possible to write down an equation for a tangent plane for $z = f(x, y)$ at the point where $(x, y) = (a, b)$.

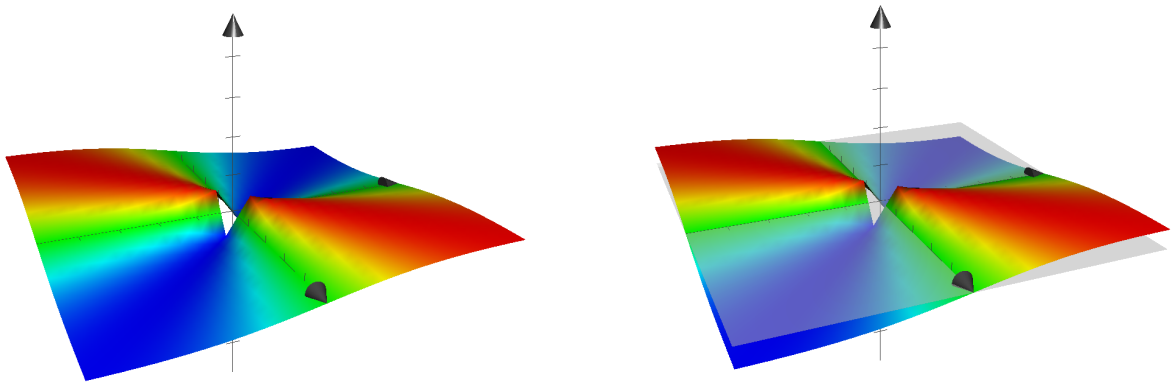
- Usually ...
- Sometimes ...

Definition. (Informal definition) A function is f is called **differentiable** at (a, b) if ...

Theorem. If f_x and f_y exist in near (a, b) and are _____ at (a, b), then f is differentiable at (a, b).

Example. Here is an example of a function that is not differentiable at $(x, y) = (0, 0)$.

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$



Question. In what sense is the tangent plane not a good approximation to the function?

Question. Is this function continuous at $(0, 0)$?

Question. What was the relationship between differentiability and continuity in Calculus 1?

Question. What is the relationship between differentiability and continuity for functions of several variables?

Recap: The tangent plane, the linearization, and the differential are related as follows:

Note. So far we have considered tangent planes for functions defined explicitly by an equation of the form $z = f(x, y)$.

How can we define tangent planes for functions that are defined implicitly by an equation of the form $F(x, y, z) = 0$? (Hint: remember level surfaces and the gradient.)

Note. The tangent plane for the surface $F(x, y, z) = 0$ is given by the equation:

Example. Find the equation for the tangent plane to the surface $xyz^2 = 6$ at the point $(3, 2, 1)$.

Note. Any surface that is defined explicitly by the equation $z = f(x, y)$ can also be defined implicitly by the equation $F(x, y, z) = 0$, where F is related to f as follows:

We now have two ways to find a tangent plane for $z = f(x, y)$:

1. Use the formula from the beginning of this class for the surface $z = f(x, y)$

2. Use the formula for the tangent plane to the level surface $F(x, y, z) = 0$, for $F(x, y, z) =$...

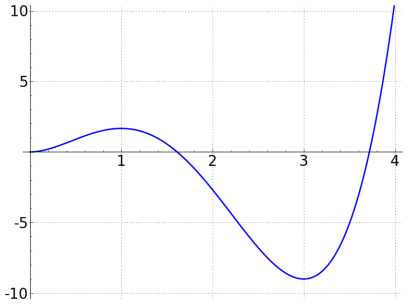
Verify that both methods give the same equation.

§15.7 Maximums and Minimums

After completing this section, students should be able to:

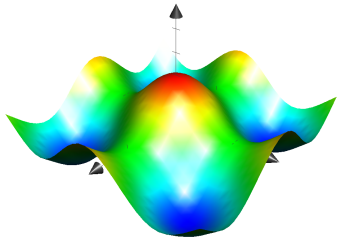
- Define maximum and minimum values and points.
- Define critical points.
- Explain the relationship between critical points and extreme points.
- Locate the critical points for a function.
- Use the Second Derivatives Test to find local maximum and minimum points and saddle points for a function.
- Give conditions for when a function is guaranteed to have an absolute maximum and minimum value.
- Find absolute maximum and minimum values for a continuous function on a closed bounded region.

Recall: For a function $f(x)$ of one variable, f has a local maximum at $x = a$ if ...



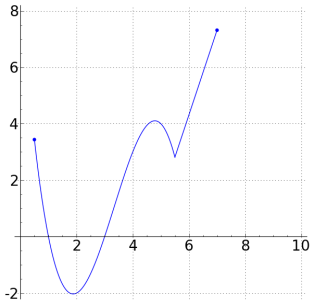
and f has a local minimum at $x = a$ if ...

Definition. A function of two variables $f(x, y)$ has a local maximum at (a, b) if ...



and $f(x, y)$ has a local minimum at (a, b) if ...

Recall: For a function $f(x)$ of one variable, a critical number is ...

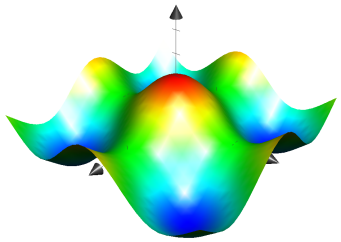


Recall: For a function $f(x)$ of one variable, max and min points are related to critical numbers.

- If $f(x)$ has a local max or min at $x = a$, then a is / is not / may or may not be a critical number.

- If a is a critical number, then $f(x)$ does / does not / may or may not have a local max or min at a.

Definition. For a function $f(x, y)$ of two variables, the point (a, b) is a **critical point** if



Max and min points are related to critical points:

- If $f(x, y)$ has a local max or min at (a, b), then (a, b) is / is not / may or may not be a critical point.

- If (a, b) is a critical point, then $f(x, y)$ does / does not / may or may not have a local max or min at (a, b).

Recall: The 2nd derivative test for functions of one variable:

Suppose $f''(x)$ is continuous in an open interval around a. Suppose $f'(a) = 0$.

If $f''(a) > 0$, then f has a local _____ at a.

If $f''(a) < 0$, then f has a local _____ at a.

Definition. The **discriminant** of f at (a, b) is given by

$$D =$$

Theorem. (Second Derivatives Test) Suppose that the second partial derivatives of f exist and are continuous on a disk around (a, b), and suppose that $f_x(a, b) = f_y(a, b) = 0$. Then

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then f has a _____ at (a, b).

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then f has a _____ at (a, b).

(c) If $D < 0$ then f has a _____ at (a, b).

(d) If $D = 0$, then _____ .

Example. Find the local maxes and mins and saddles for

$$f(x, y) = x^2 + y^2 + x^2y + 4$$

.

END OF VIDEOS

Question. What is the difference between a local maximum point and an absolute maximum point?

Question. A critical point for $f(x, y)$ is a point (x_0, y_0) such that (select all that apply)

A. Either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ DNE, or else either one or the other is 0.

B. Either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ DNE, or else both of them are 0.

C. The tangent plane to $z = f(x, y)$ doesn't exist at (x_0, y_0), or else it is horizontal.

Question. If (x_0, y_0) is a critical point for the function $f(x, y)$, then:

A. f must have a local max or min at (x_0, y_0)

B. f cannot have a local max or min at (x_0, y_0)

C. f may or may not have a local max or min at (x_0, y_0).

Question. If f has a local maximum value at (x_0, y_0), then:

A. f must have a critical point at (x_0, y_0)

B. f cannot have a critical point at (x_0, y_0)

C. f may or may not have a critical point at (x_0, y_0).

Question. If a DIFFERENTIABLE function f has a maximum or minimum value at (x_0, y_0), then (select all that are true)

A. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

B. $D_{\vec{u}} f(x_0, y_0) = 0$ for all unit vectors $\vec{u}$.

C. The tangent plane to f at (x_0, y_0) is horizontal.

D. $\nabla f(x_0, y_0) = 0$

Question. Suppose $f(x, y)$ has continuous first and second partial derivatives and (x_0, y_0) is a critical point. If $f_{xx}(x_0, y_0) > 0$, then

A. f has a local max at (x_0, y_0)

B. f has a local min at (x_0, y_0)

C. f has a saddle at (x_0, y_0)

D. There is not enough information to determine the behavior of f at (x_0, y_0)

Theorem. (Second Derivatives Test) Suppose that the second partial derivatives of f exist and are continuous on a disk around (a, b), and suppose that $f_x(a, b) = f_y(a, b) = 0$. Then

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then f has a _____ at (a, b).

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then f has a _____ at (a, b).

(c) If $D < 0$ then f has a _____ at (a, b).

(d) If $D = 0$, then _____ .

Example. Find the local maximum points, local minimum points, and saddle points for $f(x, y) = x^3 - 3x + 3xy^2$.

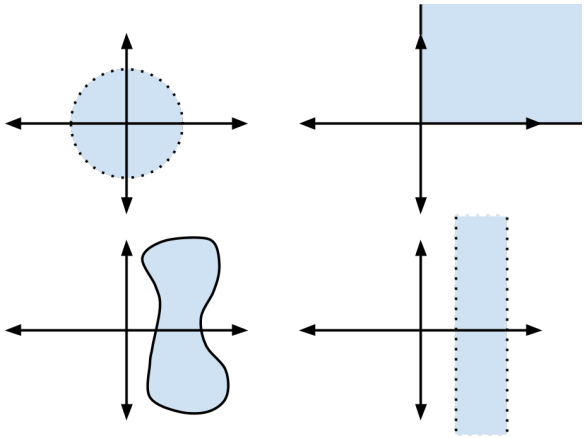
Example. Find the point on the plane $2x - y + z = 3$ that is closest to the point $(0, 1, 1)$.

Note. Recall, in Calc 1:

- If f is continuous and $[a, b]$ is a closed interval, then $f(x)$ (circle one) must / may or may not achieve an absolute maximum and minimum value on $[a, b]$.
- The abs. maximum value of f on $[a, b]$ occurs (circle one) at one of the end points a or b / at a critical point / either at a critical point or at one of the end points a or b.

Question. How do we generalize the idea of a closed interval to $\mathbb{R}^2$?

On which of these would you expect a continuous function to always have a max and a min?



Definition. (Informal definition) A **closed** set is one that ...

Definition. A **bounded** set is one that ...

Theorem. (Extreme Value Theorem) If $f(x, y)$ is _____ on a _____, _____ set D, then f attains its absolute max and abs min value.

The absolute max and min occur either at _____ or on _____.

Note. To find the abs max and min of f on D

1.

2.

3.

Example. Find the abs max and min of $f(x, y) = x^2 - 2xy + 2y$ on the triangular region with vertices $(0, 2)$, $(3, 0)$, $(3, 2)$.

Example. Maximize $f(x, y) = 2x^3 + y^4$ on the disk $D = \{(x, y) | x^2 + y^2 \leq 1\}$.

Example. Find the absolute maximum and absolute minimum of the function $f(x, y) = 3x^2 - 2y^2 - 4y$ on the region bounded by the curves $y = x^2$ and $y = 3$.

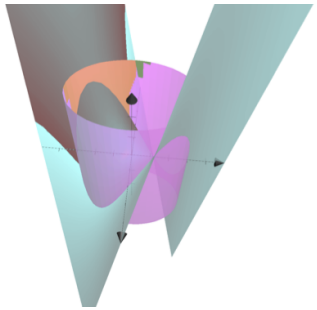
§15.8 Lagrange Multipliers

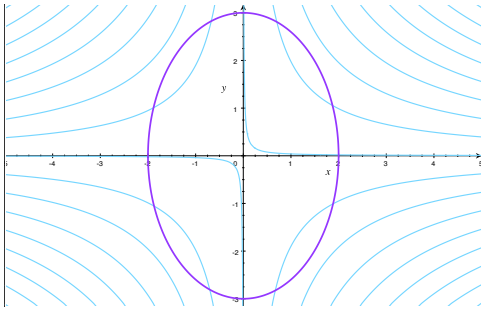
After completing this section, students should be able to:

- explain what type of problem Lagrange Multipliers can be used for
- give a qualitative explanation of why the maximum value of a function $f(x, y)$ subject to a constraint condition $g(x, y) = k$ occurs where ∇f is parallel to ∇g.
- use Lagrange multipliers to solve constrained optimization problems
- use Lagrange multipliers to maximize a function on the boundary of a region

Example. Find the rectangle of largest area that can be inscribed in the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ (with sides parallel to the x and y axes)

Consider the graph of $f(x, y) = 4xy$ and the graph of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ in 3-dimensions (on the left).





Consider the graph of the level curves of $f(x, y) = 4xy$ and the graph of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ in 2-dimensions (on the right).

Theorem. (Lagrange multipliers) To find the max and min values of $f(x, y)$ subject to the constraint $g(x, y) = k$ (assuming these extreme values exist and $\nabla g \neq 0$ on the constraint curve $g(x, y) = k$), we need to:

1. find all values (x_0, y_0) such that ∇f is parallel to ∇g, i.e. where $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$.
2. compare the size of $f(x_0, y_0)$ on all these candidate points.

Note. In practice, we set up the equations:

and solve this system of equations for x, y, and λ.

Note. For this to work, g must be "smooth" (have continuous g_x and g_y) Otherwise we must also check "corners" and "cusps" where g_x or g_y don't exist or have discontinuities.

Note. Method of Lagrange multipliers also works to maximize or minimize functions $f(x, y, z)$ of 3 variables, subject to a constraint surface $g(x, y, z) = k$.

We set

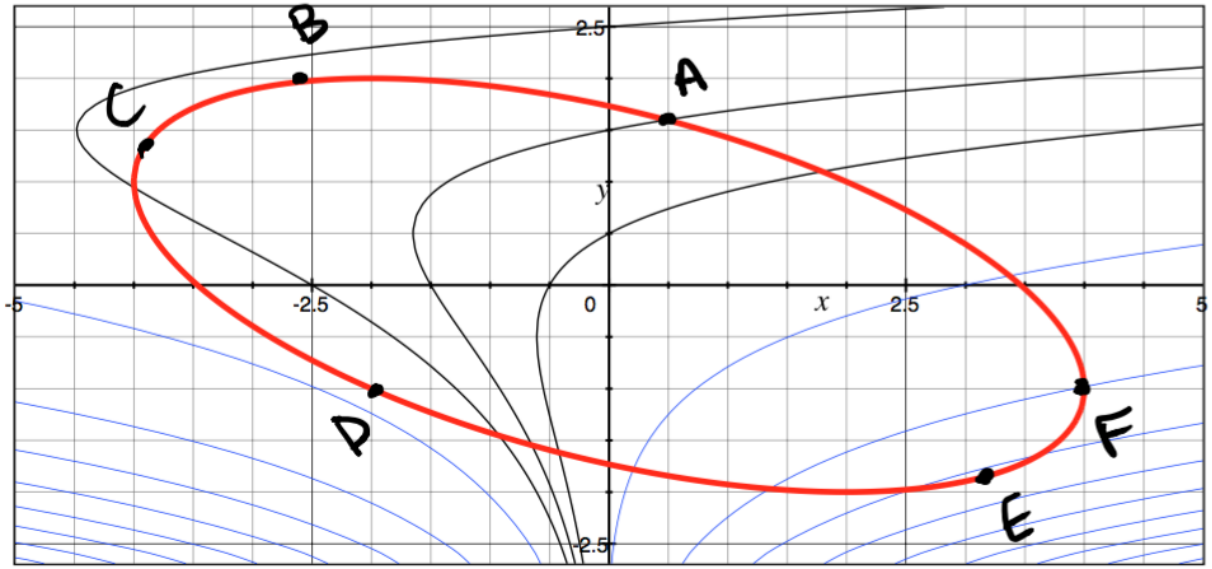
$$f_x(x, y, z) = \lambda g_x(x, y, z)$$
$$f_y(x, y, z) = \lambda g_y(x, y, z)$$
$$f_z(x, y, z) = \lambda g_z(x, y, z)$$
$$g(x, y, z) = k$$

and solve this system of equations for x, y, z, and λ.

Example. Find the rectangle of largest area that can be inscribed in the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, using Lagrange multipliers.

END OF VIDEO

Example. The graph below depicts the level curves of the function $f(x, y)$ in thin blue and black curves, and a path $g(x, y) = 5$ with a thick red curve. Where would you expect to find the maximum value of $f(x, y)$ subject to the constraint that $g(x, y) = 5$? Why?



The idea behind Lagrange multipliers is that the maximum of $z = f(x, y)$ subject to the constraint $g(x, y) = k$ (for some constant k) has to occur where ∇f is _____ to ∇g.

Example. Find the point on the plane $2x - y + z = 3$ that is closest to the point $(0, 1, 1)$.

Example. Maximize $f(x, y) = 2x^3 + y^4$ on the disk $D = \{(x, y) | x^2 + y^2 \leq 1\}$.

Extra Example. A lidless cardboard box is to be made with a volume of 4 cubic meters. Find the dimensions of the box that requires the least amount of cardboard.

Extra Example. Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 - 2y + 1$ on the semicircular region $R = \{(x, y) | x^2 + y^2 \leq 4, y \geq 0\}$