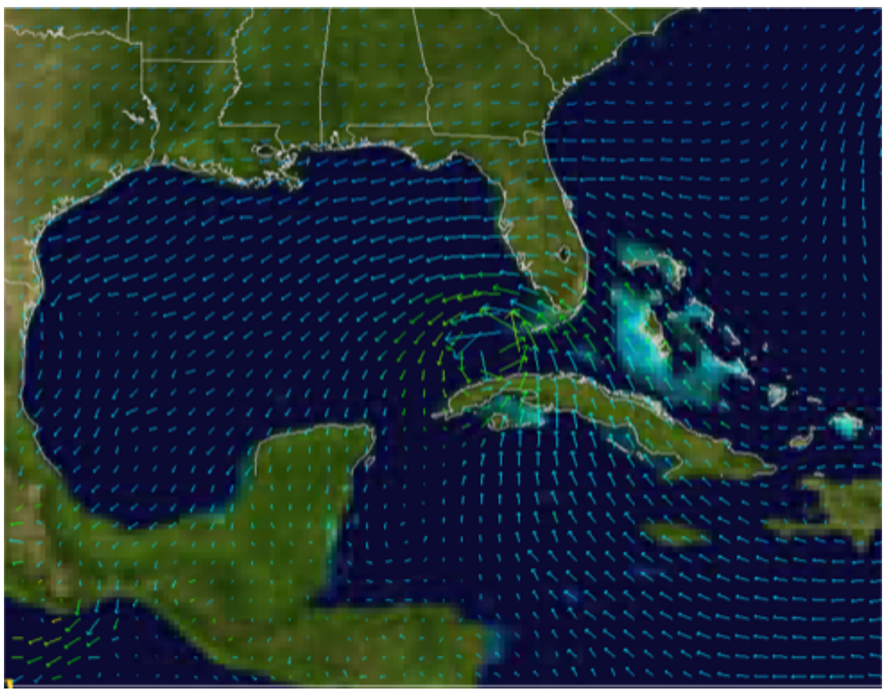
§17.1 Vector Fields

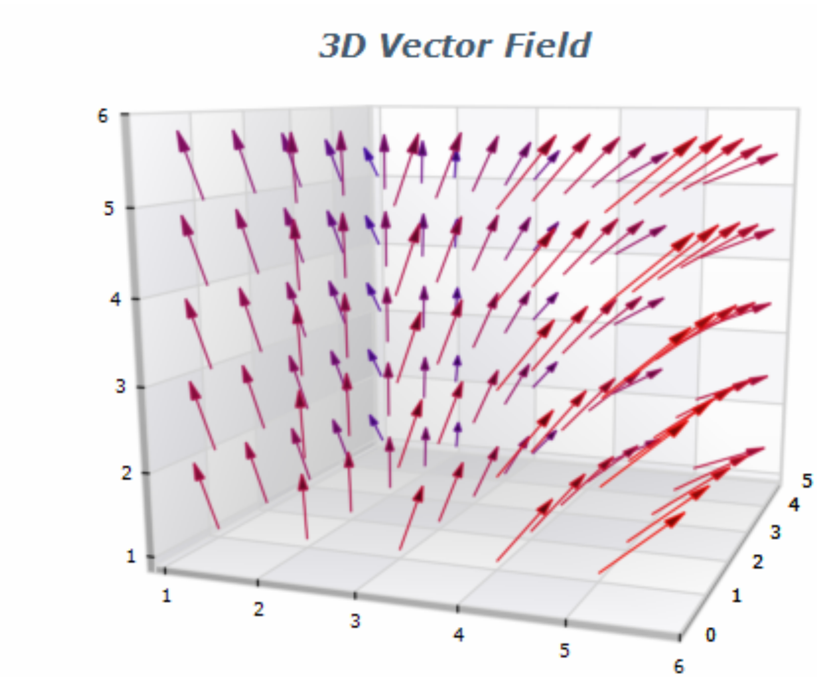
After completing this section, students should be able to:

- Define a vector field on $\mathbb{R}^2$ or $\mathbb{R}^3$.
- Match equations of vector fields with graphical representations of the vector fields.
- Define a conservative vector field (gradient field).
- Give an example of a vector field that is not conservative.

Definition. A **vector field** on $\mathbb{R}^2$ is a function ...



Definition. A **vector field** on $\mathbb{R}^3$ is a function ...



Example. Represent $\vec{F}(x, y) = y\vec{i} + x\vec{j}$ by drawing some of the vectors.

END OF VIDEO

Review. Which of these is a vector field on either $\mathbb{R}^2$ or $\mathbb{R}^3$?

A. $\vec{F}(x, y, z) = xy\vec{i} + z\vec{j} - 2\vec{k}$

B. ∇f, where $f(x, y) = 5x^2 + y$

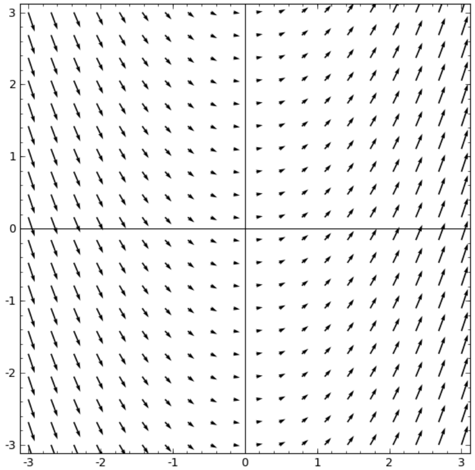
C. $F(x, y, z) = 4 - x^2 - y^2 + z$

D. $x = t\cos(t)$, $y = t\sin(t)$

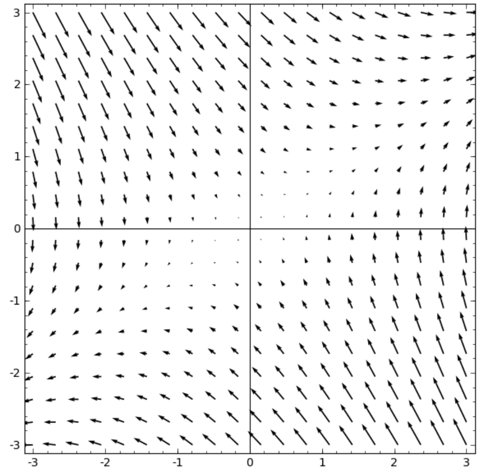
E. $\vec{r}(t) =< 2t, 3 - t, t^2 >$

Example. Draw the vector field $\vec{F}(x, y) = x\vec{i} + \vec{j}$ by drawing some vectors.

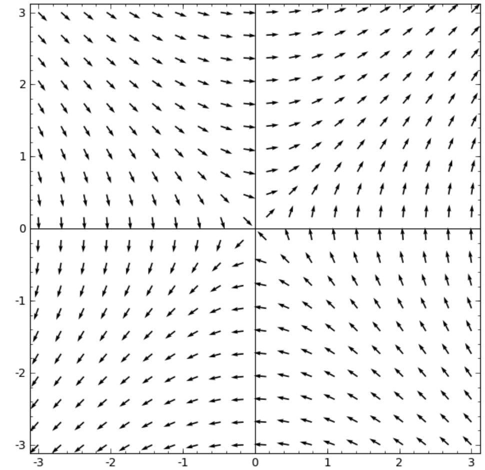
Example. Match the vector field with the plot.



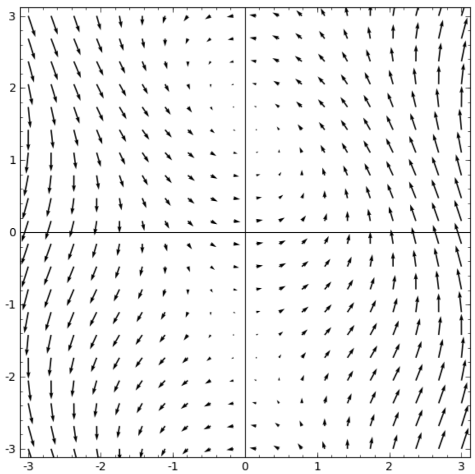
A)



B)



C)



D)

1) $\vec{F}(x, y) = \cos(x + y)\vec{i} + x\vec{j}$

2) $\vec{G}(x, y) = \vec{i} + x\vec{j}$

3) $\vec{H}(x, y) = y\vec{i} + (x - y)\vec{j}$

4) $\vec{J}(x, y) = \dfrac{y}{\sqrt{x^2 + y^2}}\vec{i} + \dfrac{x}{\sqrt{x^2 + y^2}}\vec{j}$

Example. The force of gravity can be written as a vector field. Let

M = mass of earth
m = mass of object
G = gravitational constant
$\vec{F}$ = force of gravity

Newton's law says $||\vec{F}|| = \dfrac{mMG}{r^2}$, where r is the distance between the object and the center of the earth.

Write $\vec{F}$ as a vector field. Hint: put the origin at the center of the earth.

A vector field $\vec{F}$ is a **radial vector field** if ...

that is,

$F(x, y) =$

Question. Is the force of gravity a radial vector field?

Definition. A vector field $\vec{F}$ is **conservative** if it is the gradient of some function. That is, there exists a function f such that $\vec{F} = \nabla f$.

Definition. If $\vec{F} = \nabla f$, then f is called the ________________ for $\vec{F}$.

Question. Is the force of gravity a conservative vector field?

Definition. The **equipotential curves / surfaces** of a potential function are ...

Question. What are the equipotential surfaces of the potential function for the force of gravity?

Question. What can you say about the vectors of a conservative vector field $\vec{F}$ and the equipotential curves of its potential function ϕ?

Question. Are there any vector fields that aren't conservative? Give an example.

§17.2 Line Integrals

After completing this section, students should be able to:

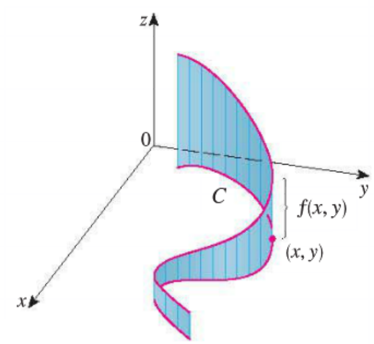
- Define and compute a line integral of a function over a curve with respect to arclength, for a function with input in $\mathbb{R}^2$ or R^3 and output in $\mathbb{R}$.
- Define and compute a line integral of a function over a curve with respect to dx or dy (or dz for functions of three variables), for a function with input in $\mathbb{R}^2$ or R^3 and output in $\mathbb{R}$.
- Define and compute a line integral of a vector field over a curve, for a vector field with input in $\mathbb{R}^2$ and output vectors in $\mathbb{R}^2$, or for a vector field with input in $\mathbb{R}^3$ and output vectors in $\mathbb{R}^3$.
- Explain how reparametrizing curves does or does not affect the value of these line integrals.

In this section, we will work with parametrized curves:

we will only consider curves that are **smooth**, which means:

or **piecewise smooth**, which means:

We want to define $\int_C f(x,y)\,ds$, the integral of a function of 2 variables over a curve "with respect to arclength".



Special case: If $f(x,y) = 1$, then $\int_C f(x,y)\,ds$ should equal ...

Definition. The line integral of f over C *with respect to arc length* is given by:

$$\int_C f(x,y)\,ds =$$

Note. It is also possible to define $\int_C f(x, y)\, ds$ as the limit of a Riemann sum, which is equivalent.

Note. The line integral of f over C with respect to arc length does not depend on the parametrization of C as long as the parametrization goes from $t = a$ to $t = b$ **with** $\mathbf{a} < \mathbf{b}$
.

Example. Calculate $\int_C xy\, ds$, for $C : x = t^2, y = 2t, 0 \leq t \leq 1$.

Definition. The line integral of f *with respect to* x is

$$\int_C f(x,y)\,dx =$$

The line integral of f *with respect to* y is

$$\int_C f(x,y)\,dy =$$

The line integral

$$\int_C f(x,y)\,dx + g(x,y)\,dy =$$

Note. The line integral with respect to x and the line integral with respect to y are independent of parametrization **provided that the parametrizations traverse** C **in the same direction**.

Example. Find $\int_C y\,dx + x\,dy$ where C is the line segment from $(1, -1)$ to $(4, 3)$.

END OF VIDEOS

Question. Let C be a smooth curve parametrized as $\vec{r}(t) =< x(t), y(t) >$ for $a \le t \le b$. The line integral of $f(x, y)$ over the curve C with respect to *arclength* can be written as: (select all that apply)

A. $\int_C f(x, y)\, ds$

B. $\int_C f(x, y)dx + f(x, y)dy$

C. $\int_{t=a}^{b} f(x(t), y(t))dt$

D. $\int_{t=a}^{b} f(x(t), y(t)) \sqrt{(x(t))^2 + (y(t))^2}\, dt$

E. $\int_{t=a}^{b} f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

F. $\int_{t=a}^{b} f(x(t), y(t))\, x'(t)\, dt + f(x(t), y(t))\, y'(t)\, dt$

Example. Calculate $\int_C x \, ds$, where C is the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.
.

Example. Find $\int_C x^2\,dx + y^2\,dy$ where C consists of the arc of the circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, 2)$ followed by the line segment from $(0, 2)$ to $(4, 3)$.

Extra Example. Set up the line integral for the following problem:

A thin wire has the shape of the first quadrant part of the circle with center the origin and radius a. If the density function is $\rho(x, y) = kxy$, find the mass of the wire.

The third and last kind of line integrals: the line integral of a vector field over a curve.

We want to define $\int_C \vec{F}\, d\text{something}$, and get a scalar answer.

Definition. For a vector field $\vec{F}(x, y)$ over a curve C parametrized by $\vec{r}(t) =< x(t), y(t) >$, define the line integral of $\vec{F}$ over C by:

$$\int_C \vec{F} \circ \ d\vec{r} =$$

Equivalently, this can be written in terms of the components of $\vec{r}(t)$.

Equivalently, if we also write $\vec{F}$ in components: $\vec{F}(x, y) =< f(x, y), g(x, y) >$ ...

Equivalently, this can be written in terms of a line integral of a function with respect to arc length (hint: first write it in terms of the unit tangent vector):

Motivation for this definition comes from physics concepts of work and circulation.

Work = force · distance

Find an expression for the work done by a vector field $\vec{F}(x, y)$ as it pushes a particle along a curve C.

For a vector field $\vec{F}$ that represents the velocity field of a fluid, the **circulation** of the vector field on a closed curve C is defined as ...

and the **flux** of $\vec{F}$ across C is defined as ...

Example. Find the work done by the force field

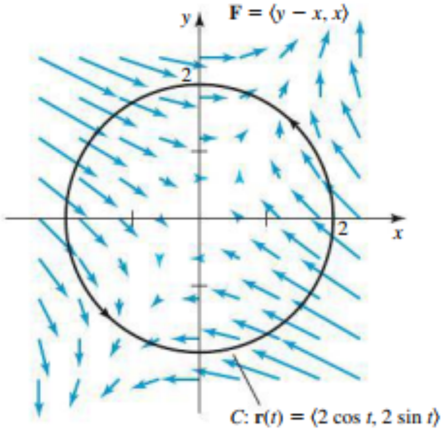
$$\vec{F}(x, y) = x\vec{i} + (y + 2)\vec{j}$$

in moving an object along an arch of the cycloid

$$\vec{r}(t) = (t - \sin t)\vec{i} + (1 - \cos t)\vec{j}$$

for $0 \leq t \leq 2\pi$.

Example. Find the circulation and flux for the vector field $\vec{F} =< y - x, x >$ and the curve $C : \vec{r}(t) =< 2\cos t, 2\sin t >$, for $0 \leq t \leq 2\pi$



There are analogous definitions for functions and vector fields of 3-variables:

For a function $f(x,y,z)$, and a curve C in $\mathbb{R}^3$ parametrized by $\vec{r}(t) =< x(t), y(t), z(t) >$,

Definition.

$$\int_C f(x,y,z)\, ds =$$

The line integral of $f(x,y,z)$ with respect to x is

$$\int_C f(x,y,z)\, dx =$$

The line integral of $f(x,y,z)$ with respect to y is

$$\int_C f(x,y,z)\, dy =$$

The line integral of $f(x,y,z)$ with respect to z is

$$\int_C f(x,y,z)\, dz =$$

The line integral

$$\int_C f(x,y,z)\, dx + g(x,y,z)\, dy + h(x,y,z)\, dz =$$

For a vector field $\vec{F}(x, y, z)$ in $\mathbb{R}^3$

$$\int_C \vec{F}(x, y, z) \circ \, d\vec{r} =$$

Note. All of these line integrals are independent of the parametrization of C up to a positive or negative sign, as will be specified below.

Example. Parametrize the curve $y = x$ from $(0, 0)$ to $(2, 2)$ in two ways

(a) going from $(0, 0)$ to $(2, 2)$

(b) going from $(2, 2)$ to $(0, 0)$.

How do $dx = x'(t)\, dt$ and $dy = y'(t)\, dt$ compare for the two different paramtrizations?

How does $ds = \sqrt{(x'(t))^2 + (y'(t))^2}$ compare for the two different parametrizations?

Question. When we parametrize a curve in the opposite direction, then which of the following change sign?

A. $\displaystyle\int_C f(x, y)\, dx$

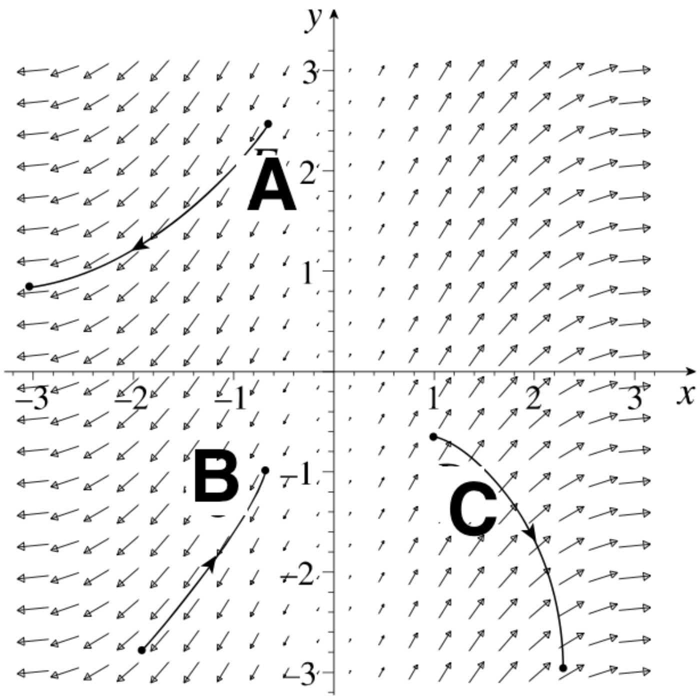
B. $\displaystyle\int_C f(x, y)\, dy$

C. $\displaystyle\int_C f(x, y)\, ds$

D. $\displaystyle\int_C \vec{F}(x, y) \circ d\vec{r}$

There is also a change in sign when we integrate "backwards" from a larger t value to a smaller t value, as usual.

Example. For which picture does $\int_C \vec{F} \circ d\vec{r}$ have the greatest value?



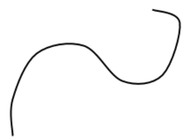
Hint: decide if $\int_C \vec{F} \circ d\vec{r}$ is positive, negative, or zero.

§17.3 Conservative Vector Fields

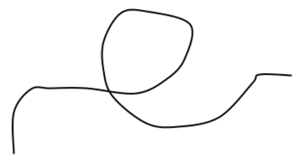
After completing this section, students should be able to:

- Give informal definitions of simple curves and closed curves and of open, connected, and simply connected regions of the plane.
- Determine whether a curve is simple and / or closed.
- Determine whether a region of the plan is open, connected, and / or simply connected.
- State the Fundamental Theorem for Line Integrals and use it to evaluate integrals of conservative vector fields the lazy way,
- Define what it means for a vector field to be conservative in terms of a gradient.
- Give an equivalent characterization of conservative in terms of independence of path for line integrals and in terms of line integrals on closed paths.
- For a vector field $\vec{F}(x, y) =< P(x, y), Q(x, y) >$, give an equivalent characterization of conservative in terms of the partial derivatives of P and Q, under appropriate conditions.
- Find the potential function for a conservative vector field in two of three dimensions.

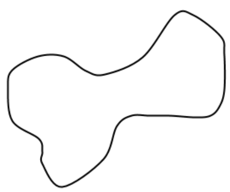
Definition. A curve C is **closed** if the initial point and terminal point of C coincide.



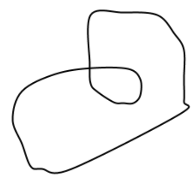
simple, not closed



not simple, not closed



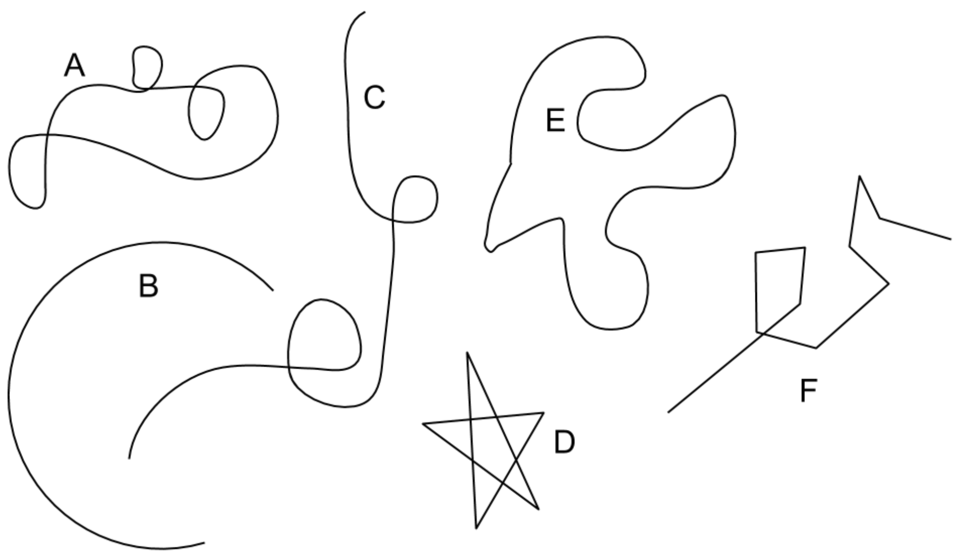
simple, closed



not simple, closed

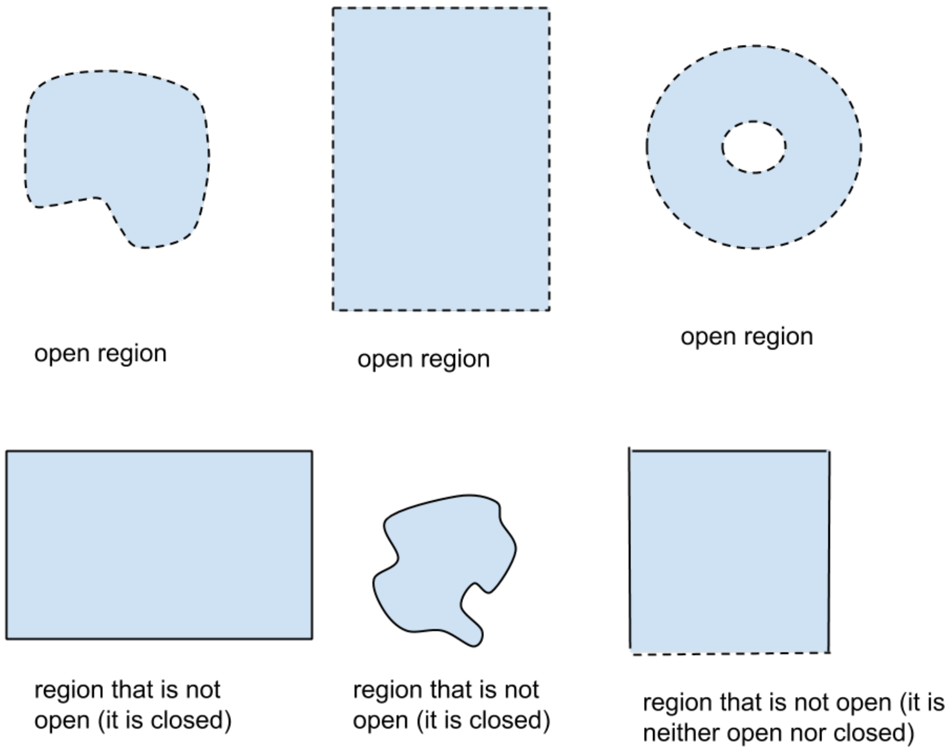
Definition. A curve C is **simple** if C does not intersect itself, except possibly at its endpoints.

Classify each curve: is it simple? closed?

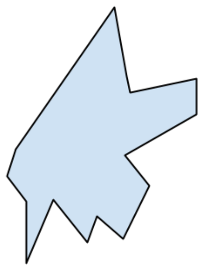


Definition. (Informal definition) A region D is **open** if it doesn't contain any of its boundary points.

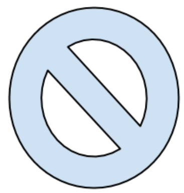
A region D is **closed** if it contains all of its boundary points.



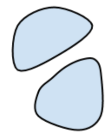
Definition. A region D is **connected** if any two points in D can be joined by a path in D.



connected region

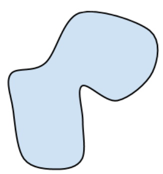


connected region



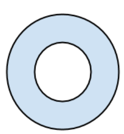
disconnected region

Definition. A region D is **simply connected** if every simple closed curve C in D, C encloses only points of D. Infomally, simply connected means ...

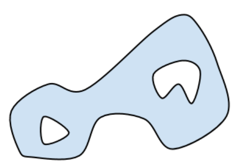


simply connected

simply connected

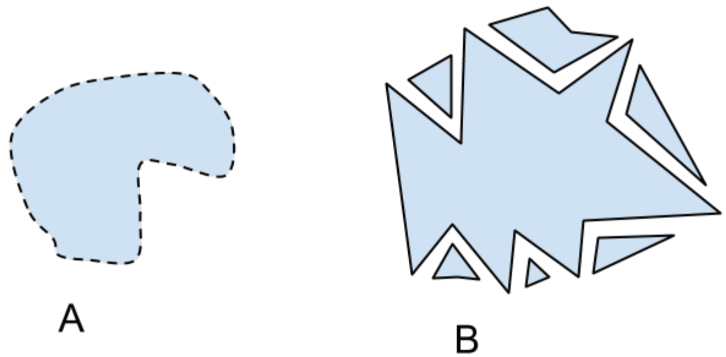


not simply connected



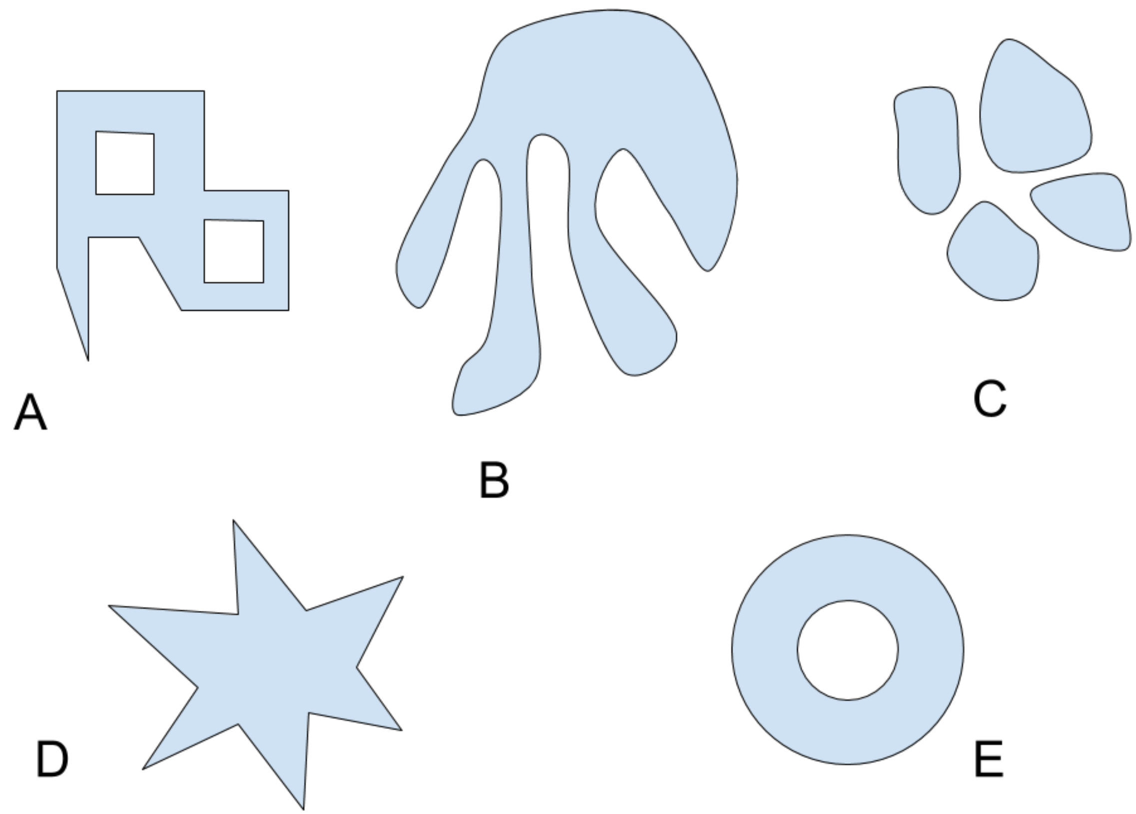
not simply connected

Classify each region: is it open? connected? simply connected?



END OF VIDEOS

Review. Which of these regions are connected? Which are simply connected?



Example. Is the vector field $\vec{F}(x, y) = < \sin(x^2), xy >$ conservative? See if you can figure out the answer by taking derivatives instead of integrals.

Theorem. Suppose $\vec{F}(x,y) =< f(x,y), g(x,y) >$ where $f(x,y)$ and $g(x,y)$ have continuous first partial derivatives,. If $\vec{F} = \nabla\phi$ for some function ϕ, then

Therefore, if $\frac{\partial f}{\partial y} \neq \frac{\partial g}{\partial x}$ on a region D, then ...

Question. Does the converse hold? Is it true that if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ on a region D, then $\vec{F} = \nabla\phi$ for some function ϕ?

Theorem. Suppose $\vec{F}(x, y) =< f(x, y), g(x, y) >$ *on an open, connected, and simply connected region*, where $f(x, y)$ and $g(x, y)$ have continuous first partial derivatives. If ...

Example. Determine if $\vec{F} = (3 + 2xy)\vec{i} + (x^2 - 3y^2)\vec{j}$ is conservative. If it is conservative, find ϕ such that $\nabla\phi = \vec{F}$.

Question. What about vector fields in $\mathbb{R}^3$?

Theorem. Suppose $\vec{F}(x, y, z) =< f(x, y, z), g(x, y, z), h(x, y, z) >$ where f, g, and h have continuous first partial derivatives. If $\vec{F} = \nabla\phi$ for some function ϕ, then ...

Theorem. Suppose $\vec{F}(x, y, z) =< f(x, y, z), g(x, y, z), h(x, y, z) >$ *on an open, connected, and simply connected region*, where f, g, and h have continuous first partial derivatives. If ...

Example. Show that $\vec{F}(x, y, z) = \sin y\,\vec{i} + (x\cos y + \cos z)\vec{j} - y\sin z\,\vec{k}$ is conservative. Find f such that $\nabla f = \vec{F}$.

For the rest of this section, we will only work with piecewise smooth curves.

Recall: in Calc 1, the Fundamental Theorem of Calculus (FTC) says:

$$\int_a^b F'(x)\, dx =$$

For line integrals, we have a similar theorem where ________________ plays the role of F'.

Theorem. (Fundamental Theorem for Line Integrals - FTLI) Let $f(x, y)$ be a differentiable function whose gradient is continuous. Let C be a smooth curve parametrized by $\vec{r}(t)$ for $a \leq t \leq b$. Then

$$\int_C \nabla\phi \circ \, d\vec{r} =$$

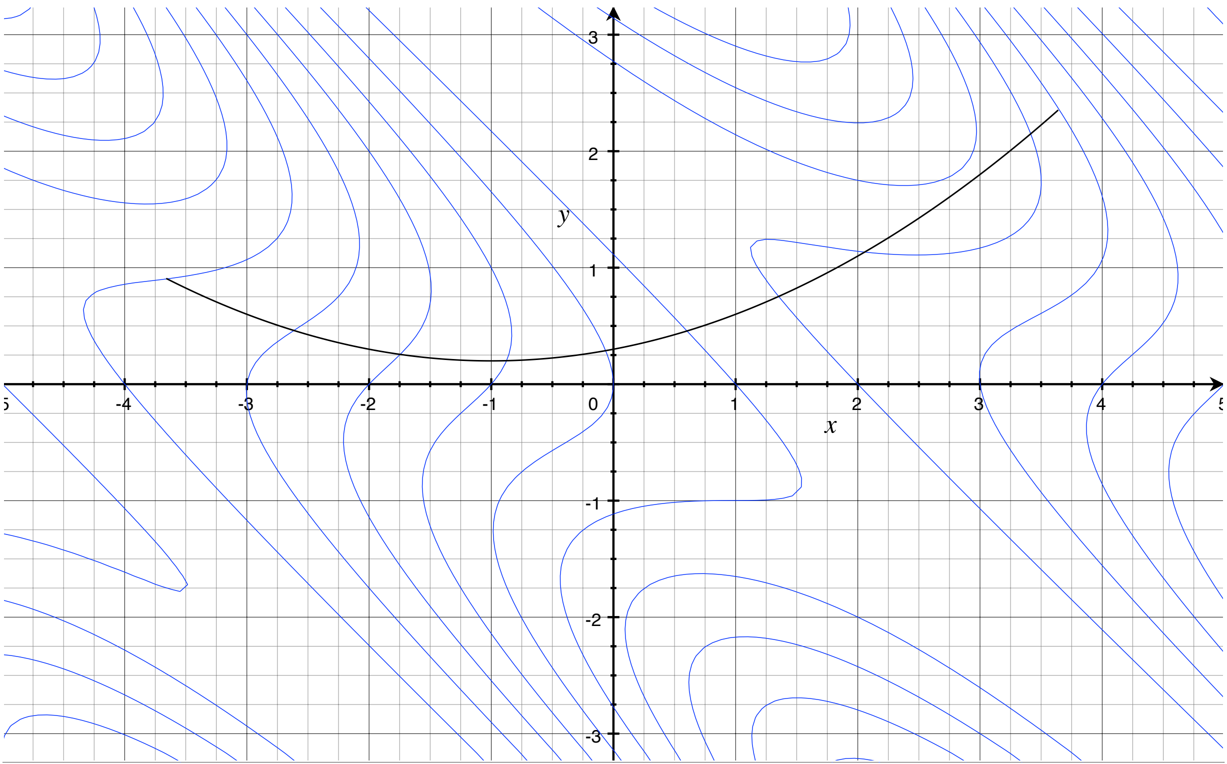
Example. Find $\int_C \nabla(xe^y) \circ d\vec{r}$ where C is a straight line from $(3, 0)$ to $(8, 1)$.

Proof. Use the chain rule in reverse:

Example. The figure shows a curve C and a contour map of a function

$$\phi(x, y) = y \sin(x + y) + x$$

whose gradient ∇f is continuous. Find $\int_C \nabla\phi \circ d\vec{r}$



Example. Find the work done by gravity to move an object of mass 100 kg from the point $(3.1, 3, 5)$ (in millions of meters) to the point $(3, 3, 5)$.

Note: the force of gravity is given by

$$\vec{F}(x, y, z) = -\frac{mMGx}{(x^2 + y^2 + z^2)^{3/2}}\vec{i} - \frac{mMGy}{(x^2 + y^2 + z^2)^{3/2}}\vec{j} - \frac{mMGz}{(x^2 + y^2 + z^2)^{3/2}}\vec{k}$$

Recall: $\vec{F} = \nabla\phi$, where $\phi =$

$M = 5.97219 \times 10^{24} kg$

$G = 6.67384 \times 10^{-11} \frac{m^3}{kg\ s^2}$

Note.

1. The Fundamental Theorem of Line Integrals (FTLI) only applies to vector fields that are *gradients*, i.e. to *conservative* vector fields.
2. The FTLI says that we can evaluate the line integral of a conservative vector field $\nabla\phi$ if we only know the value of ϕ at ______________ .
3. In the language of physics, the FTLI says that the work done in moving an object from one point to another by a conservative force (like gravity) does not depend on ______________ , only on ______________ .
4. If C_1 and C_2 are two paths connecting points A and B, then

$$\int_{C_1} \nabla\phi \circ \, d\vec{r} - \int_{C_2} \nabla\phi \circ \, d\vec{r} =$$

Definition. For continuous vector field $\vec{F}(x, y)$, we say that $\int_C \vec{F}(x, y) \circ d\vec{r}$ is independent of path if ...

... for any two curves C_1 and C_2 with the same initial and terminal end points.

True or False: If a vector field is conservative, then it has line integrals that are independent of path.

True or False: If a vector field has line integrals that are independent of path, then it is conservative.

Theorem. Suppose $\vec{F}$ is a continuous vector field on an open, connected region D. Then $\int_C \vec{F} \circ \, d\vec{r}$ is independent of path $\iff$

Another characterization of independence of path:

Theorem. Suppose $\vec{F}$ is a continuous vector field on a region D, then $\displaystyle\int_C \vec{F} \circ d\vec{r}$ is independent of path $\iff$

for every closed path $\hat{C}$ in D.

Proof:

Theorem. For a continuous vector field $\vec{F}$ on an open, connected region D, the following are equivalent:

1.

2.

3.

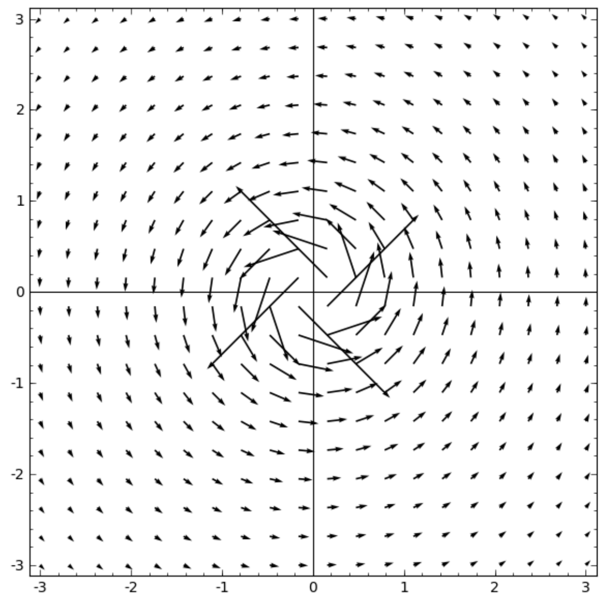
4.

Note. If the region D is not simply connected, then $f_y = g_x$ on D does not guarantee that $\vec{F}$ is conservative.

Example. Compute $\int \vec{F} \circ \, d\vec{r}$, where $\vec{F} =< ye^x + 2xyz, e^x + x^2z + 2yz, x^2y + y^2 + 3 >$ and $\vec{r}(t) =< t, t^2 - 2, \dfrac{3}{t+1} >$ from $0 \leq t \leq 2$.

Extra Example. Is this vector field conservative? Why or why not?

$$\vec{F}(x,y) = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$$



§17.4 Green's Theorem

After completing this section, students should be able to:

- Determine if boundary curves are positively oriented.
- State Green's Theorem.
- Use Green's Theorem to compute a line integral of a vector field by converting it to a double integral over a region.
- Use Green's Theorem to compute a double integral over a region by converting it to a line integral of a vector field.
- Use Green's Theorem to compute an area.
- Use Green's Theorem to prove that if $\vec{F} =< f, g >$ is a vector field on a simply connected region and $f_y = g_x$ then $\vec{F}$ is conservative.
- Give an example of a vector field $\vec{F} =< f, g >$ with a non-simply connected domain for which $f_y = g_x$ but $\vec{F}$ is not conservative.

Green's Theorem relates a line integral over simple closed curve C to a double integral over the region that C encloses.

Definition. A simple closed curve C that forms the boundary of a region D is called **positively oriented** if, as you traverse C, the region D is always to the left.

A collection of curves $C_1 \cup C_2 \cup \cdots C_n$ that together form the boundary of a region D are positively oriented if each curve is positively oriented.

Theorem. *Green's Theorem Let $\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$ be a vector field and suppose that P and Q have continuous first partial derivatives. Let ∂D be a positively oriented, piecewise smooth curve or collection of curves that bound a region D. Then*

$$\int_{\partial D} \qquad\qquad = \iint_D$$

Note. Alternative notations for $\int_{\partial D} P\,dx + Q\,dy$ are: $\int_C P\,dx + Q\,dy$ OR $\oint_C P\,dx + Q\,dy$, where $C = \partial D$.

Example. Use Green's Theorem to evaluate the line integral $\int_C ye^x\,dx + 2e^x\,dy$ where C is the positively oriented boundary of the rectangle with vertices $(0,0)$, $(3,0)$, $(3,4)$, and $(0,4)$.

END OF VIDEO

Review. Green's theorem says that $\int_C f\,dx + g\,dy = ...$

A. $\int\int_R f_x - g_y\,dA$

B. $\int\int_R g_x - f_y\,dA$

C. $\int\int_R g_y - f_x\,dA$

D. $\int\int f_y + g_x\,dA$

E. $\int\int f_x + g_y\,dA$

F. $\int\int f_y + g_x\,dA$

Where C is the positively oriented boundary of R.

Note. If $\vec{F}$ is the vector field with components f and g, then we can also write Green's theorem as ...

Example. A particle starts at the origin, moves along the x-axis to $(5, 0)$, then along the quarter-circle $x^2 + y^2 = 25$ with $x \geq 0$ and $y \geq o$, and then down the y-axis back to the origin. Use Green's Theorem to find the work done on this particle by the force field $\vec{F}(x, y) =< \sin x, \sin y + xy^2 + \frac{1}{3}x^3 >$.

We have used Green's Theorem to convert a line integral to an integral over a region. What about vice versa?

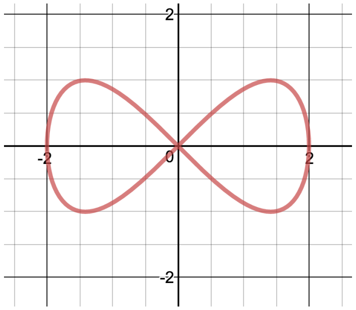
How can Green's Theorem be used to compute an area?

Since $\displaystyle\int\int_D 1\,dA = \text{area}(D)$, to use Green's Theorem, we need to find f and g such that ...

What can we use for f and g?

Note. area$(D) = \int_{\partial D}$

Example. Find the area inside the hourglass figure given by the curve $x = 2\cos(t)$, $y = \sin(2t)$.



Alternate forms of Green's Theorem

1. $\int_C \vec{F} \circ d\vec{r} = \int_C f\, dx + g\, dy = \ldots$

2. $\int_C \vec{F} \circ \vec{n}\, ds = \int_C \qquad\qquad\qquad =$

Definition. The expression $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}$ is called the ...

Definition. The expression $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y}$ is called the ...

Note. In words, the ″curl″ form of Green's theorem says ...

The ″divergence″ form of Green's theorem says ...

Note. .

1. Green's theorem works for any vector field $\vec{F}$. How is this different from the Fundamental Theorem of Line Integrals?

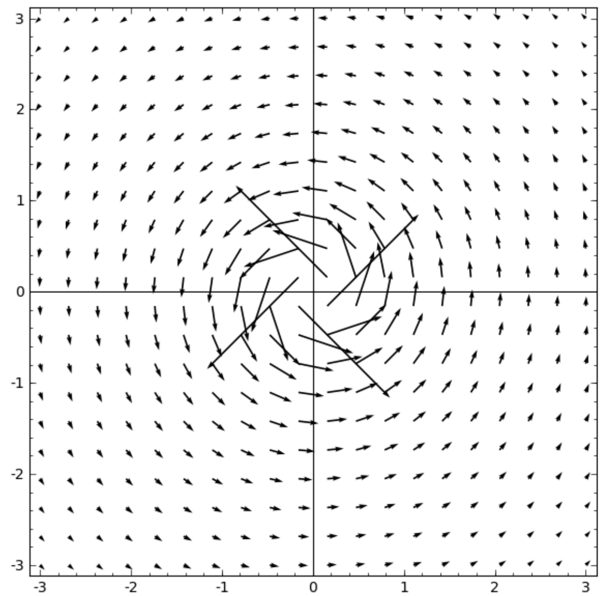
2. Green's theorem is analogous to the fundamental theorem of calculus. How?

3. Green's theorem applies only to 2-dimensional vector fields (but there is a more general theory that applies in 3 dimensions and in n dimensions).

4. Green's theorem can be used to prove that when $f_y = g_x$ on an open, simply connected region, then $\vec{F} = f\vec{i} + g\vec{j}$ is conservative.

Proof of fact from §17.3: If f and g have continuous partial derivatives on an open, simply connected region, $\vec{F} = f\vec{i} + g\vec{j}$ is conservative if and only if ...

Extra Example. If $\vec{F}(x,y) = < \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} >$, show that $\int_C \vec{F} \circ d\vec{r} = 2\pi$ for every positively oriented simple closed curve that encloses the origin.



PROOFS

Theorem. Green's Theorem Let $\vec{F}(x, y) = f(x, y)\vec{i} + g(x, y)\vec{j}$ be a vector field and suppose that f and g have continuous first partial derivatives. Let ∂D be a positively oriented, piecewise smooth curve or collection of curves that bound a region D. Then

$$\int_{\partial D} f\, dx + g\, dy = \int\int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

Proof. .

Step 1: Prove that if D is a Type I region, then $\displaystyle\int_{\partial D} f\, dx = \int\int_D \left(-\frac{\partial f}{\partial y}\right) dA.$

Step 2: Prove that if D is a Type II region, then $\displaystyle\int_{\partial D} g\, dy = \int\int_D \left(\frac{\partial g}{\partial x}\right) dA.$

Step 3: Prove Green's Theorem for regions that are both Type I and Type II .

Step 4: Prove Green's Theorem in general.

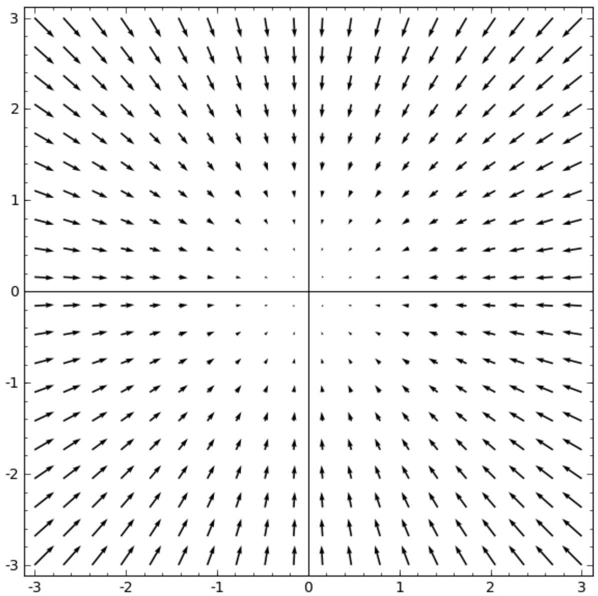
§17.5 Divergence and Curl

After completing this sections, students should be able to:

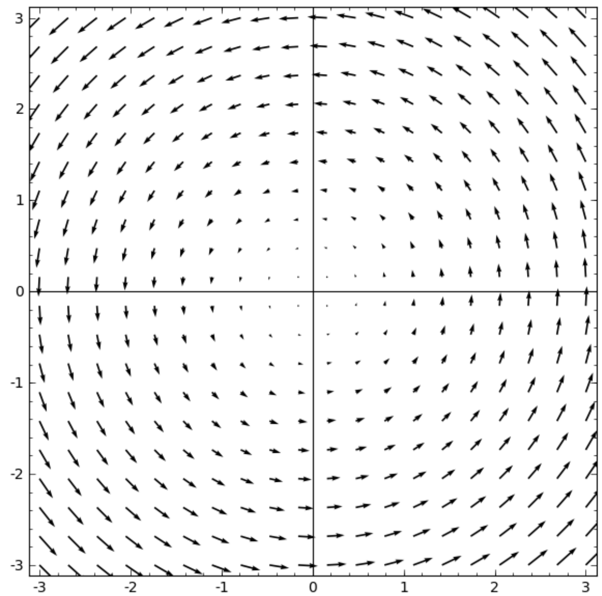
- Use a plot of a vector field to estimate if the divergence at a given point is positive, negative, or zero from a plot of the vector ficled.
- Use a plot of a vector field to estimate whether the curl of a vector field is the zero vector or not, and if it is not, estimate the direction of the curl at a point.
- Compute the divergence and curl of a vector field from its equation.
- Determine if a vector field on a simply connected region of $\mathbb{R}^3$ is conservative by computing its curl.
- Determine if a vector field could be the curl of another vector field by computing its divergence.

Note. Divergence (div) applies to vectors in 3-dimensions.

Definition. (Informal definition) If $\vec{F}$ represents the velocity of a fluid, then div $\vec{F}(x, y, z)$ represents ...



$\vec{F} = < -x, -y, 0 >$



$\vec{F} = < -2y, 2x, 0 >$

Definition. For $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, the divergence div $\vec{F}$ is defined as:

Note that div $\vec{F}$ is a (circle one) scalar / vector.

Example. Compute the divergence of the previous 2 examples:

1. $\vec{F}(x,y,z) =< -x, -y, 0 >$

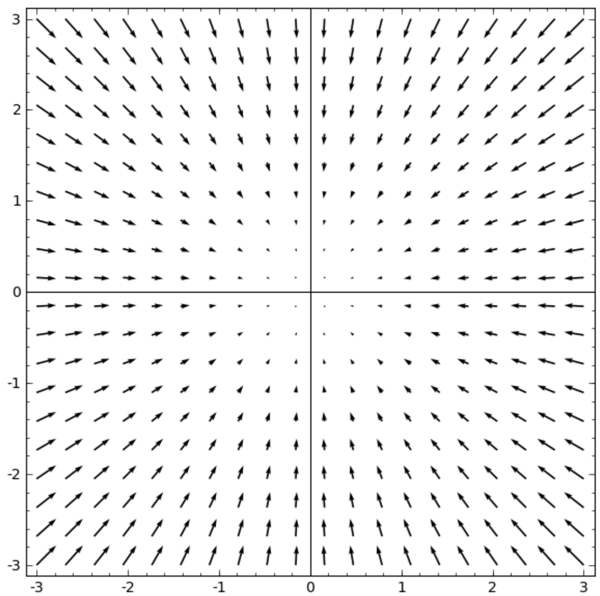
2. $\vec{F}(x,y,z) =< -2y, 2x, 0 >$

Note. Curl applies to vectors in 3 dimensions.

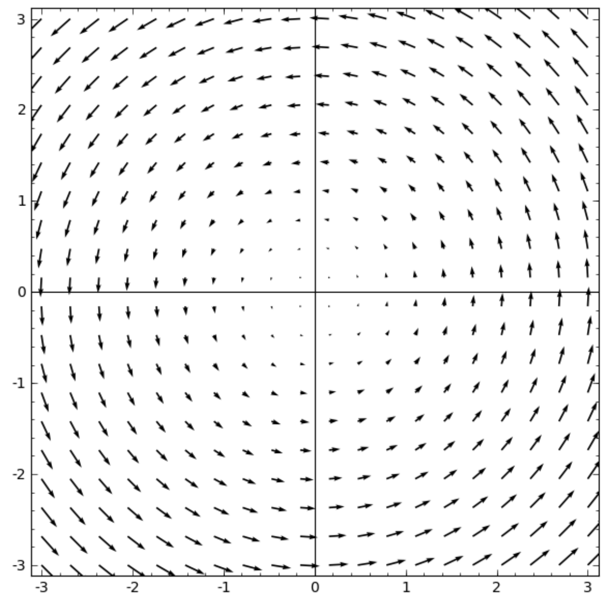
Definition. (Informal definition) If $\vec{F}$ represents the velocity of a fluid, then curl $\vec{F}(x, y, z)$ represents ...

the direction of the curl is ...

the magnitude of the curl is ...



$\vec{F}(x, y, z) = < -x, -y, 0 >$



$\vec{F}(x, y, z) = < -2y, 2x, 0 >$

Definition. For $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, the curl $\vec{F}$ is defined as:

Note that curl $\vec{F}$ is a (circle one) scalar / vector.

Example. Compute the curl of the previous 2 examples:

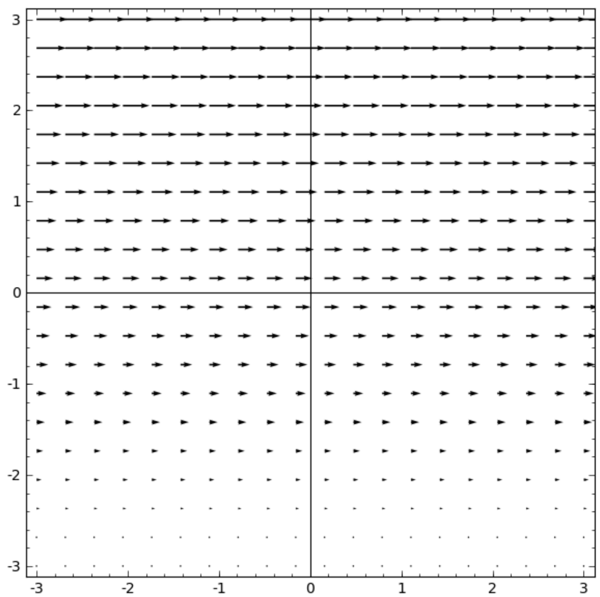
1. $\vec{F}(x, y, z) =< -x, -y, 0 >$

2. $\vec{F}(x, y, z) =< -2y, 2x, 0 >$

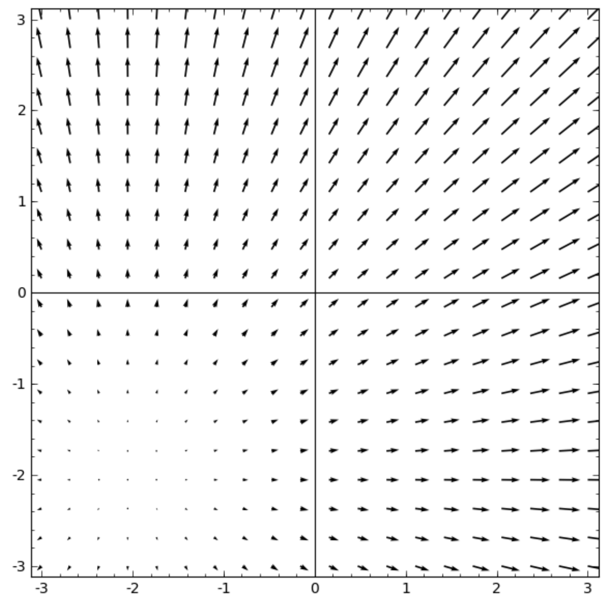
END OF VIDEOS

Note. Divergence (div) and curl apply to vectors in 3-dimensions.

Definition. (Informal definition) If $\vec{F}$ represents the velocity of a fluid, then div $\vec{F}(x, y, z)$ represents ...



$\vec{F} = < \frac{y}{3} + 1, 0, 0 >$



$\vec{F} = < x + 2, y + 2, 0 >$

Example. The vector field $\vec{F} = f\vec{i} + g\vec{j} + h\vec{k}$ is shown in the xy plane and looks the same in all other horizontal planes. Is div $\vec{F}$ positive, negative, or zero at the origin?

Definition. For $\vec{F} = f\vec{i} + g\vec{j} + h\vec{k}$, the divergence div $\vec{F}$ is defined as:

Example. Compute the divergence of the previous 2 examples:

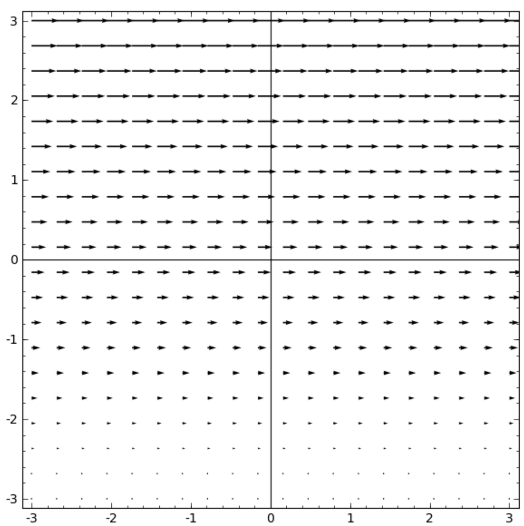
1. $\vec{F}(x, y, z) =< \frac{y}{3} + 1, 0, 0 >$

2. $\vec{F}(x, y, z) =< x + 2, y + 2, 0 >$

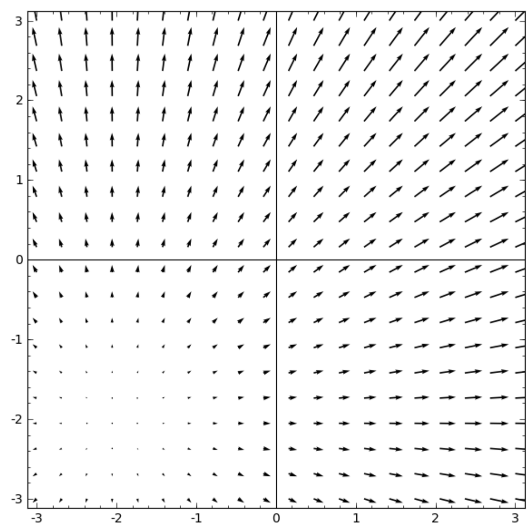
Definition. (Informal definition) If $\vec{F}$ represents the velocity of a fluid, then curl $\vec{F}(x, y, z)$ represents ...

The direction of the curl is ...

and the magnitude of the curl is ...



$\vec{F}(x, y, z) = < \frac{y}{3} + 1, 0, 0 >$



$\vec{F}(x, y, z) = < x + 2, y + 2, 0 >$

Example. The vector field $\vec{F} = f\vec{i} + g\vec{j} + h\vec{k}$ is shown in the xy plane and looks the same in all other horizontal planes. Determine whether curl $\vec{F} = \vec{0}$ at the origin. If not, in which direction does curl $\vec{F}$ point?

Definition. For $\vec{F} = f\vec{i} + g\vec{j} + h\vec{k}$, the curl $\vec{F}$ is defined as:

Example. Compute the curl of the previous 2 examples:

1. $\vec{F}(x, y, z) =< \frac{y}{3} + 1, 0, 0 >$

2. $\vec{F}(x, y, z) =< x + 2, y + 2, 0 >$

Question. Is div $\vec{F}$ a vector or a scalar?

Is curl $\vec{F}$ a vector or a scalar?

Example. For the vector field $\vec{F}(x, y, z) = xyz\,\vec{i} + 4x^2y + z^2\,\vec{j} + xy^3z\,\vec{k}$.

1. Find $div(\vec{F})$
2. Find $curl(\vec{F})$
3. Find $div(curl(\vec{F}))$

Theorem. If $\vec{F} = f\vec{i} + g\vec{j} + h\vec{k}$ is a vector field on $\mathbb{R}^3$ and f, g, and h have continuous second-order partial derivatives, then

$$\text{div curl } \vec{F} = \underline{\hspace{3cm}}$$

Proof:

Example. Find the curl of $\vec{F} = < \dfrac{y}{x} + 1, \ln(x) + z^2, 2yz >$

Is $\vec{F}$ conservative?

§17.6 Surface Integrals

After completing this section, students should be able to:

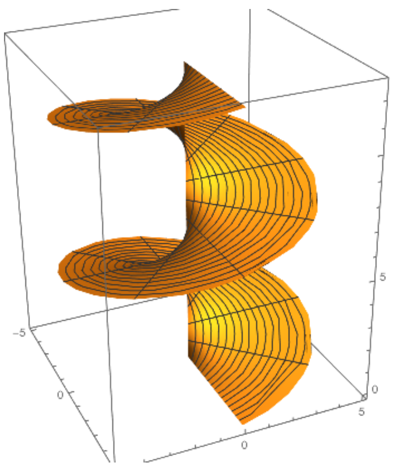
- Parametrize a surface.
- Find the tangent plane to a parametrized surface.
- Compute the surface area of a surface.
- Give an informal explanation of where the formula for surface area of a parametrized surface comes from.
- Give an alternative formula for surface area when the surface is of the form $z = f(x, y)$ and explain how this alternative formula comes from the general formula for any parametrized surface.
- Give an alternative formula for surface area when the surface is a surface of revolution and explain how this alternative formula comes from the general formula.
- Describe what is meant by an orientation of a surface.
- Distinguish a positive orientation from a negative orientation for a closed surface.
- Compute the surface integral of a function over a surface.
- Compute the surface integral of a vector field over a surface.

Recall: A parametric curve $\vec{r}(t) = < x(t), y(t), z(t) >$ gives the x, y, and z coordinates of points on the curve in terms of another variable t.

Definition. A parametric surface $\vec{r}(u,v) = < x(u,v), y(u,v), z(u,v) >$ gives the x, y, and z coordinates of points on the surface in terms of two parameters u and v.

Example. $\vec{r}(u,v) = < u\cos(v), u\sin(v), v >$. for $0 \leq u \leq 5$, $0 \leq v \leq 4\pi$ is a parametric surface.

What curves do we get if we fix v to be a constant?

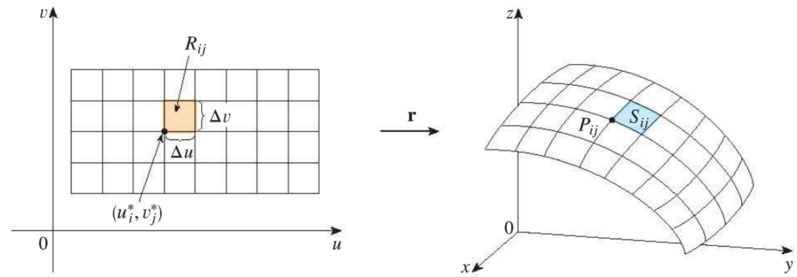


What if we fix u to be a constant?

Example. Parametrize the paraboloid $x = y^2 + z^2$.

Example. Parametrize the hemisphere $x^2 + y^2 + z^2 = 9$ with $y \geq 0$.

Goal: Find the surface area of a parametric surface $\vec{r}(u,v) = < x(u,v), y(u,v), z(u,v) >$



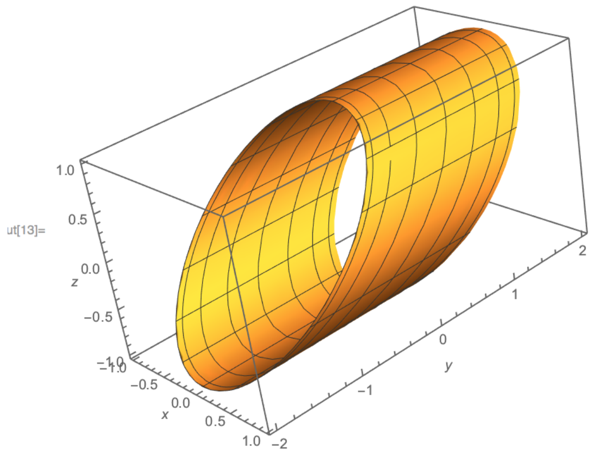
Example. Find the area of the portion of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

END OF VIDEOS

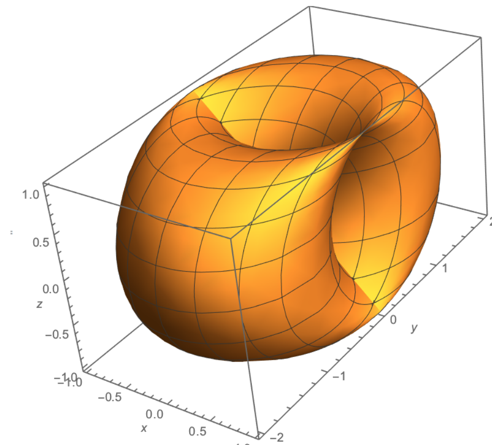
Example. Which graph represents the surface

$$\vec{r}(u,v) = < \cos(u), \sin(u) + \cos(v), \sin(v) >$$

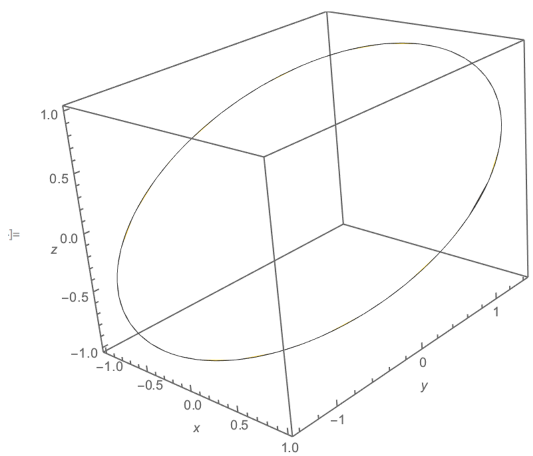
for $0 \le u \le 2\pi$, $0 \le v \le 2\pi$?



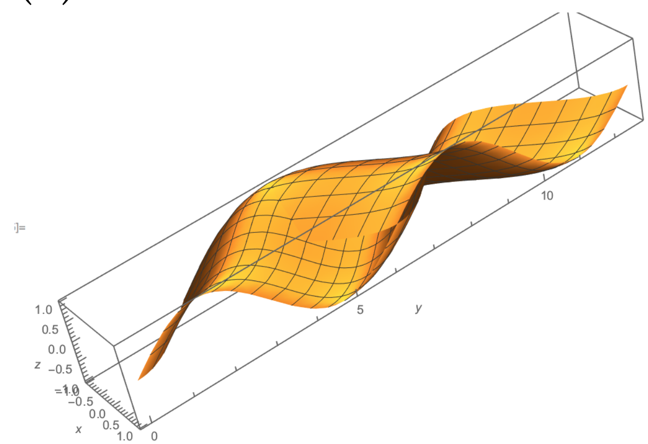
(A)



(B)



(C)



(D)

Example. How would you parametrize the following curves?

(a) $y = x^2$

(b) $x^2 + y^2 = 17$

Example. Parametrize the plane $3x + 5y + 2z = 6$

Example. Parametrize the cylinder $x^2 + z^2 = 16$ for $-5 \leq y \leq 5$.

Example. Parametrize the cap of the sphere $x^2 + y^2 + z^2 = 9$ that lies above the plane $z = 2$.

Review. The surface area of a surface with parametrization $\vec{r}(u,v)$ for $a \leq u \leq b$ and $c \leq v \leq d$ is given by:

A. $\displaystyle\int_{v=c}^{d} \vec{r}(u,v)\, du\, dv$

B. $\displaystyle\int_{v=c}^{d} \vec{r}_u \circ \vec{r}_v\, du\, dv$

C. $\displaystyle\int_{v=c}^{d} \int_{u=a}^{b} \|\vec{r}_u \times \vec{r}_v\|\, du\, dv$

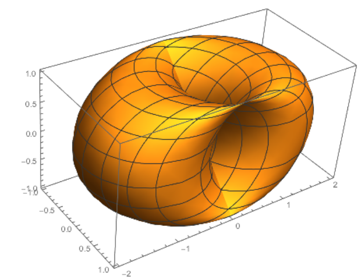
D. $\displaystyle\int_{v=c}^{d} \int_{u=a}^{b} \vec{r}_u \times \vec{r}_v\, du\, dv$

Example. Find the surface area of the cap of the sphere $x^2 + y^2 + z^2 = 9$ that lies above the plane $z = 2$.

Example. SET UP the integral to compute the surface area for the surface

$$\vec{r}(u,v) =< \cos(u), \sin(u) + \cos(v), \sin(v) >$$

for $0 \leq u \leq 2\pi$, $0 \leq v \leq 2\pi$.



Note. There are alternative formulas for surface area in special cases.

1. If the surface is of the form $z = f(x, y)$, then surface area is given by:

2. If the surface is a surface of revolution, formed by rotating the curve $y = g(x)$ around the x-axis for $a \leq x \leq b$, then surface area is given by:

Both of these formulas can be derived from the general formula for the area of a parametric surface.

A **surface integral** is analogous to a line integral.

Review. The line integral of $f(x, y)$ with respect to arclength over the curve C is defined as:

$$\int_C f(x, y, z)ds =$$

and represents:

Definition. The surface integral of $f(x, y, z)$ over the parametrized surface S is defined as:

$$\int\int_S f(x, y, z)dS =$$

and represents:

Example. Set up an integral to find $\int\int_S (x+y+z)\, dS$, where S is the part of the cylinder $x^2 + z^2 = 9$ that lies between the planes $y = 0$ and $y = 2$.

Note. If the surface is described as $z = g(x, y)$ for some function g, then

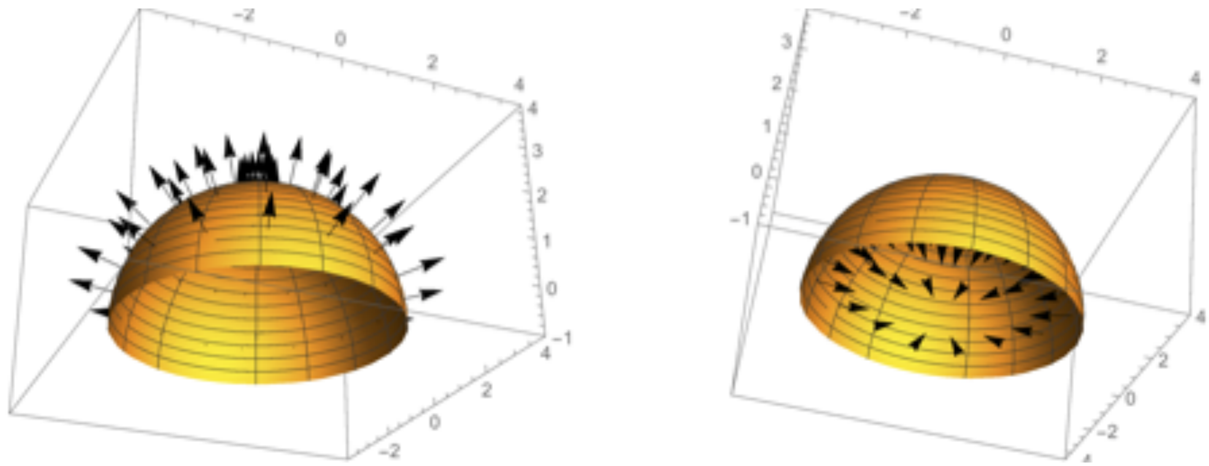
$$\int\int_S f(x, y, z)dS =$$

Example. Find the surface integral $\int\int_S x^2 + y^2 \, dS$ where S is the part of the sideways cone $y^2 + z^2 = x^2$ that lies between the planes $x = 1$ and $x = 3$, in the first octant.

Surface Integrals of Vector Fields - Preliminaries:

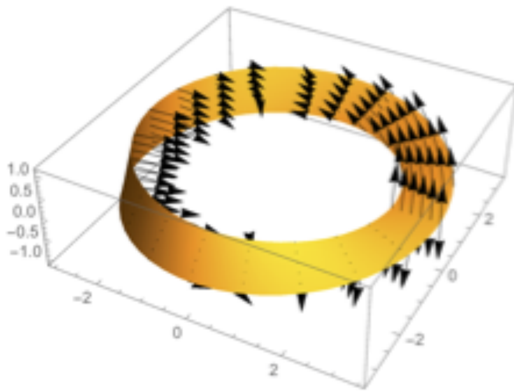
Definition. An orientation of a surface S in $\mathbb{R}^3$ is a choice of unit normal vector $\vec{n}(x, y, z)$ at every point $(x, y, z) \in S$, in such a way that $\vec{n}$ varies continuously.

Example. A hemisphere has two orientations:

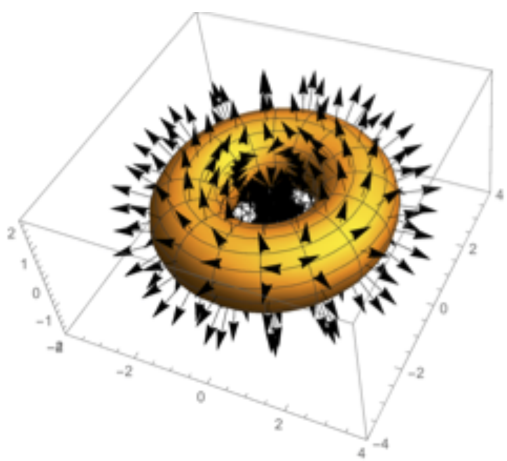
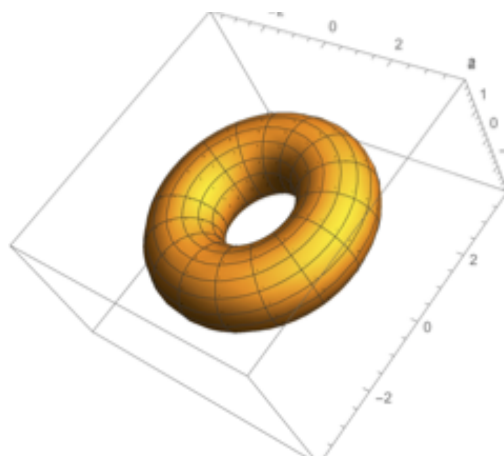


Definition. A surface is orientable if it is possible to find an orientation for it.

Question. Are all surfaces orientable?



Note. A surface in $\mathbb{R}^3$ is orientable if and only if it has "two sides". Any orientable surface has two orientations.

Definition. For a *closed surface* in $\mathbb{R}^3$ (finite with no boundary), the *positive orientation* is the orientation for which the normal vectors point ________. The *negative orientation* is the orientation for which the normal vectors point ________.

Review. The flux integral of a vector field $\vec{F}(x, y, z)$ is defined as

$$\int_C \vec{F} \circ \vec{n}\, ds =$$

and represents:

Definition. The flux integral of a vector field $\vec{F}(x, y, z)$ over a parametrized surface S is:

$$\int\int_S \vec{F}(x, y, z) \circ \ d\vec{S} =$$

This represents:

Note. The surface integral of $\vec{F}(x, y, z)$ over the surface S can be calculated as:

$$\int\int_S \vec{F}(x, y, z) \circ \ d\vec{S} =$$

Example. Set up an integral to find $\int\int_S \vec{F} \circ d\vec{S}$, where $\vec{F}(x, y, z) = z\vec{i} + y\vec{j} + x\vec{k}$ and S is the helicoid: $\vec{r}(u, v) =< u\cos(v), u\sin(v), v >$ with $0 \leq u \leq 1$, $0 \leq v \leq \pi$, oriented upward.

Question. Are surface integrals well-defined?

If we use two different parametrizations of a surface, will we still get the same answer for the surface integral?

A. No, using different parametrizations will change the values of the surface integrals in random, unpredictable ways.

B. The values of the surface integrals will stay the same up to a $\pm$ sign. Both $\int\int_S f(x, y, z)dS$ and $\int\int_S \vec{F}(x, y, z) \circ d\vec{S}$ can change sign with different parametrizations.

C. $\int\int_S f(x, y, z)dS$ will always stay the same. $\int\int_S \vec{F}(x, y, z) \circ d\vec{S}$ will stay the same as long as the normal vector $r_u \times r_v$ still points in the same direction; otherwise, it changes sign.

Note. If the surface is described as $z = s(x, y)$ and $\vec{F} = f\vec{i} + g\vec{j} + h\vec{k}$ for $(x, y) \in D$, then

$$\int\int_S \vec{F}(x, y, z) \circ \, d\vec{S} = \pm \int\int_D -f\frac{\partial s}{\partial x} - g\frac{\partial s}{\partial y} + h \, dA$$

Proof:

Example. Compute $\displaystyle\int\int_S \vec{F} \circ d\vec{S}$, where $\vec{F} = -y\vec{j} + z\vec{k}$ and S consists of the paraboloid $z = x^2 + y^2$, $0 \leq z \leq 1$, together with the disk $x^2 + y^2 \leq 1$, $z = 1$.

Example. Gauss's Law says that the net charge enclosed by a closed surface S is

$$q = \epsilon_0 \int\int_S \vec{E} \circ \, d\vec{S}$$

where ϵ_0 is a constant.

Use Gauss's Law to find the charge enclosed by the cube with vertices $(\pm 1, \pm 1, \pm 1)$ if the electric field is $\vec{E}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$.

Why does $\int\int_S \vec{F} \circ \vec{n}\, dS$ represent flux (rate of fluid flow)?

§17.7 and S17.8 Stokes' Theorem and Divergence Theorem Mini-Lesson

Review. Green's Theorem says that for $\vec{F} =< f, g >$ (assuming f, g have continuous partials and ∂D is piecewise smooth)

$$\int_{\partial D} f\, dx + g\, dy = \int \int_D \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\, dA$$

.

This can also be written in terms of curl as ...

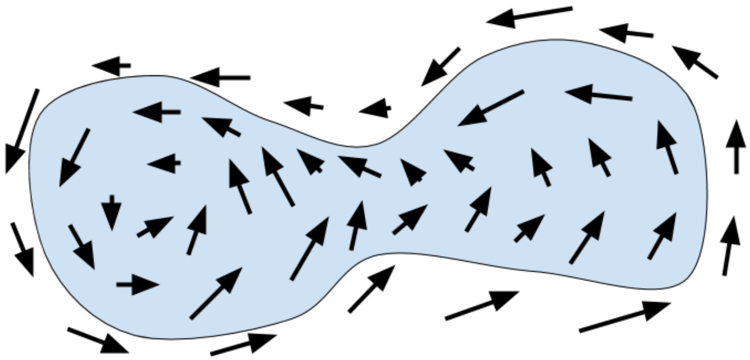
Theorem. (Stoke's Theorem) Let S be an oriented piecewise smooth surface bounded by a simple, closed, piecewise smooth boundary curve ∂S with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives in an open region of $\mathbb{R}^3$ around S. Then

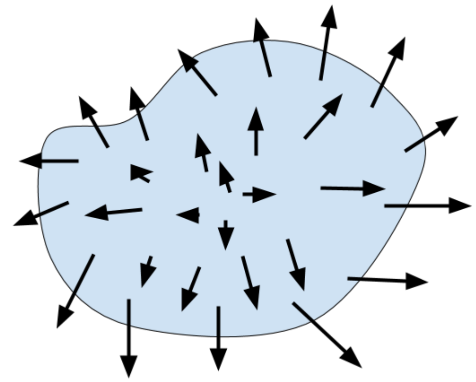
.

This can also be written as:

In words this means:

Intuition:





The Divergence form of Green's Theorem says that for $\vec{F} =< f, g >$ (assuming f, g have continuous partials and ∂D is piecewise smooth)

$$\int_{\partial D} f\, dy - g\, dx = \int \int_D \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\, dA$$

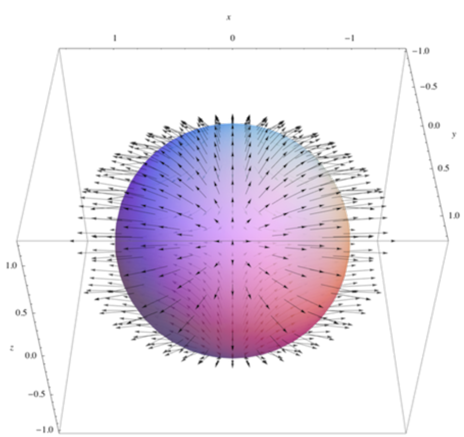
.

This can also be written in terms of divergence as:

Theorem. (Divergence Theorem) Let E be a solid region and let S be the boundary surface of E, given positive (outward) orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

In words this means:

Intuition



§16.8 Stokes Theorem

After completing this section, students should be able to:

- State Stokes Theorem in words and in symbols.
- Compare the Stokes Theorem to Green's Theorem.
- "Verify" Stokes Theorem for a given vector field and surface, by computing both $\int_{\partial S} \vec{F} \circ d\vec{r}$ and $\int \int_S curl(\vec{F}) \circ d\vec{S}$ and checking that they are equal.
- Use Stokes Theorem to compute line integrals of vector fields by rewriting them as surface integrals.
- Use Stokes Theorem to compute surface integrals by rewriting them as line integrals.

Preliminaries: If S is an oriented surface with boundary ∂S, a **positive orientation** on ∂S means that if you walk around ∂S with your head in the direction of the normal vector for S, the surface S lies ...

Review. Green's Theorem says that for $\vec{F} = < P, Q >$ (assuming P, Q have continuous partials and ∂D is piecewise smooth)

$$\int_{\partial D} P\,dx + Q\,dy = \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

.

This can also be written in terms of curl as ...

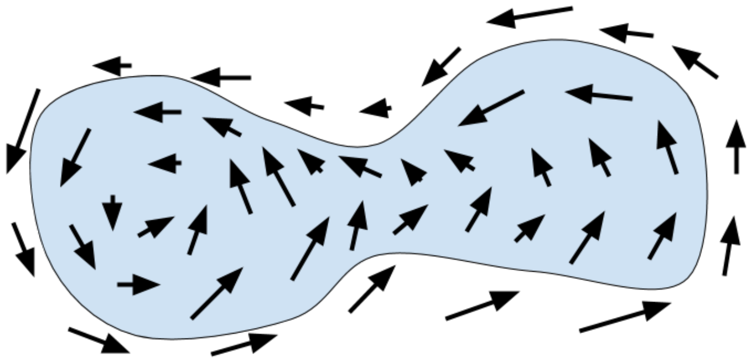
Theorem. (Stoke's Theorem) Let S be an oriented piecewise smooth surface bounded by a simple, closed, piecewise smooth boundary curve ∂S with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives in an open region of $\mathbb{R}^3$ around S. Then

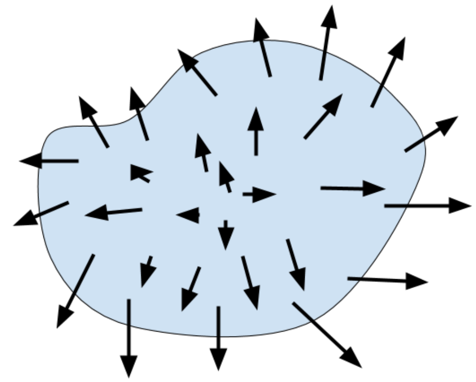
.

This can also be written as

In words this means:

Intuition:





Example. Verify Stokes theorem for the vector field $\vec{F} = -y\vec{i} + x\vec{j} - 2\vec{k}$ and S is the cone $z^2 = x^2 + y^2$ for $0 \leq z \leq 4$, oriented downwards.

Consequences of Stoke's Theorem:

1. Stokes Theorem implies: If $curl(\vec{F}) = \vec{0}$ on a simply connected region, then $\vec{F}$ is conservative.

2. Stokes Theorem implies: $\int\int_{S_1} curl(\vec{F}) \circ d\vec{S} = \int\int_{S_2} curl(\vec{F}) \circ d\vec{S}$ if S_1 and S_2 are two surfaces with the same boundary.

3. Stokes Theorem implies: Green's Theorem.

Example. Find $\int_C \vec{F} \circ d\vec{r}$ where $\vec{F}(x, y, z) = -y^2\vec{i} + x\vec{j} + z^2\vec{k}$ where C is the intersection curve of the plane $y + z = 2$ and $x^2 + y^2 = 1$.

Extra Example. Use Stokes' Theorem to evaluate $\int\int_S curl(\vec{F}) \circ d\vec{S}$, where $\vec{F}(x, y, z) = e^{xy}\vec{i} + e^{xz}\vec{j} + x^2z\vec{k}$ and S is the half of the ellipsoid $4x^2 + y^2 + 4z^2 = 4$ that lies to the right of the xz-plane, oriented in the direction of the positive y axis.

§16.9 Divergence Theorem

After completing this section, students should be able to:

- State Divergence Theorem in words and in symbols.
- Compare the Divergence Theorem to Green's theorem and Stokes' Theorem.
- Use the Divergence Theorem to compute surface integrals by rewriting them as triple integrals.

Review. Green's Theorem says that for $\vec{F} =< P, Q >$ (assuming P, Q have continuous partials and ∂D is piecewise smooth)

$$\int_{\partial D} P\,dx + Q\,dy = \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

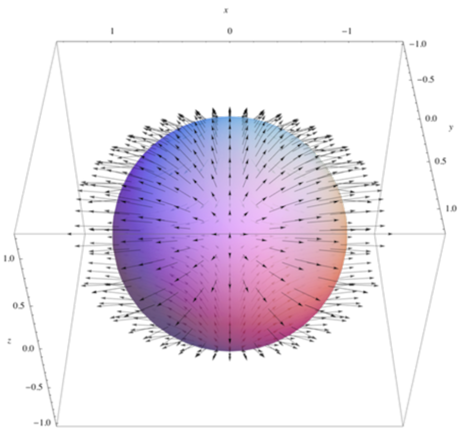
.

This can also be written as.

Theorem. (Divergence Theorem) Let E be a solid region and let S be the boundary surface of E, given positive (outward) orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

In words this means:

Intuition



Example. Verify the Divergence Theorem for the vector field $\vec{F}(x, y, z) = x^2yz\vec{i} + xy^2z\vec{j} + xyz^2\vec{k}$ over the surface that is the box bounded by the coordinate planes and the planes $x = a$, $y = b$, $z = c$.

Example. Find the flux of the vector field $\vec{F}(x, y, z) = z\vec{i} + y\vec{j} + x\vec{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

Example. Find the flux of $\vec{F}$ across the surface S, where $\vec{F} = 3xy^2\vec{i} + xe^x\vec{j} + z^3\vec{k}$ and S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$.

Extra Example. Verify the Divergence Theorem for $\vec{F} = y^2z^3\vec{i} + 2yz\vec{j} + 4z^2\vec{k}$ and E is the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 0$.

Extra Example. Consider the electric field $\vec{E}(x, y, z) = \dfrac{\epsilon Q}{||\vec{x}||^3}\vec{x}$, where the electric charge Q is located at the origin and $\vec{x} =< x, y, z >$ is a position vector. Use the Divergence Theorem to show that the electric flux of $\vec{E}$ through any closed surface S that encloses the origin is $\int\int_S \vec{E} \circ d\vec{S} = 4\pi\epsilon Q$.

Review for Final Exam

What are some of the main themes of the class, topics that have come up repeatedly?

Recurring themes of the class:

1. Tangent Planes
2. Gradient
3. Changing perspective to compute an integral
4. Parametrizing stuff
5. Different types of integrals
6. How have we defined orientation?
7. Applications
8. Visualization

What are the different ways we have computed tangent planes?

What are the ways we have used the gradient?

In what ways have we changed perspective in order to compute an integral?

How have we parametrized curves and surfaces?

What types of integrals have we defined and computed?

How have we defined orientation?

What applications or interpretations of concepts have we studied?

What techniques have we used to visualize stuff?