S26 Knots

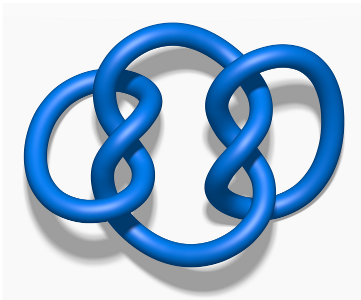
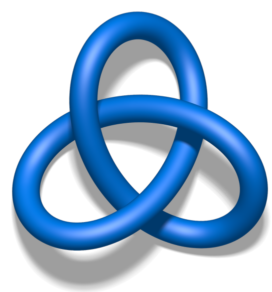
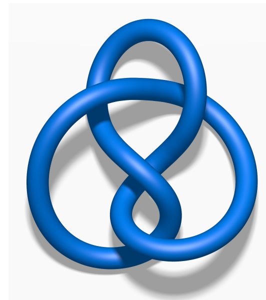
References:

- *A Decade of the Berkeley Math Circle, Volume 2*, Maia Averett's chapter on Knots.
- *The Knot Book* by Colin Adams, Chapter 1 and Chapter 6
- *Beginning Topoplogy* by Sue Goodman, Chapter 7

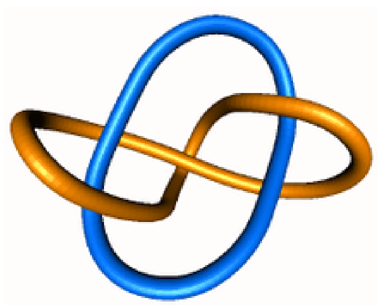
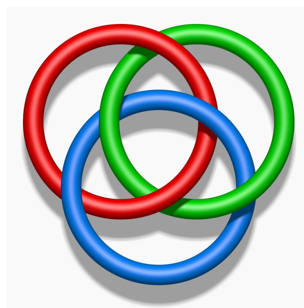
Supplies:

- power cords
- rope, string

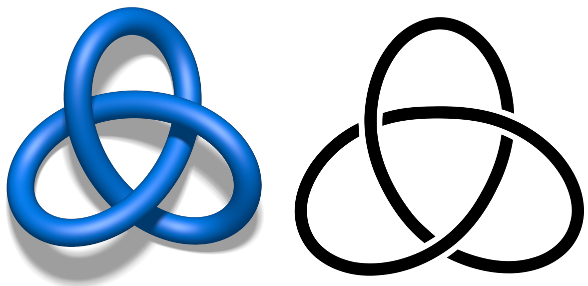
Loosely speaking, a knot is a piece of string that has been tangled up and then had the ends fused together.

When the knot is made from two or more loops of string, it is called a link. Here are some examples of links.

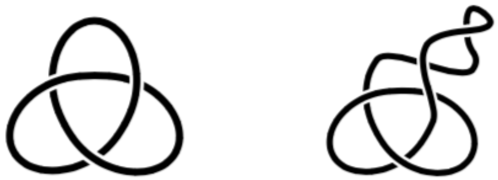
Knots can be represented by ”knot diagrams”. For example, the knot below at left can be represented by this knot diagram at right, where the gaps represent undercrossings.



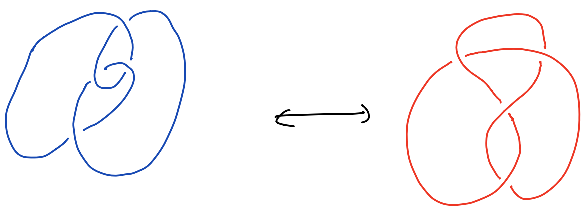
Here are some general questions we will explore:

- When should we consider two knots to be the same?
- How can we tell different knots apart?
- How can we tell if a knot is knotted or not?

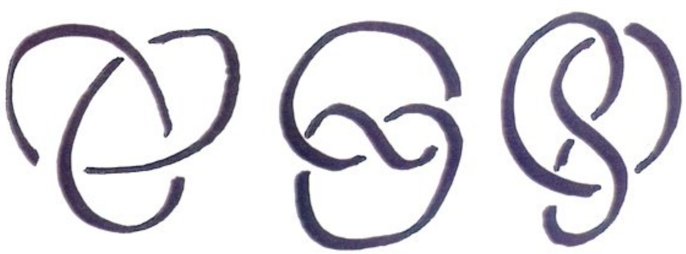
1. In what sense are the following knots equivalent?



2. Are these knots equivalent?

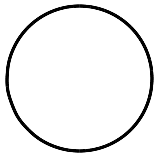


3. Which of these knots, if any, are equivalent?

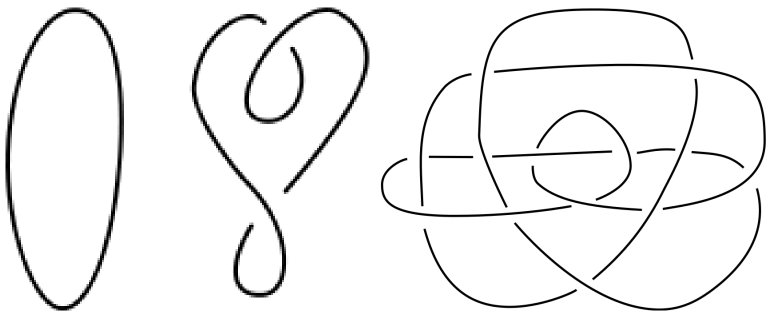


(a) (b) (c)

Definition: An *unknot* is a knot equivalent to this one:

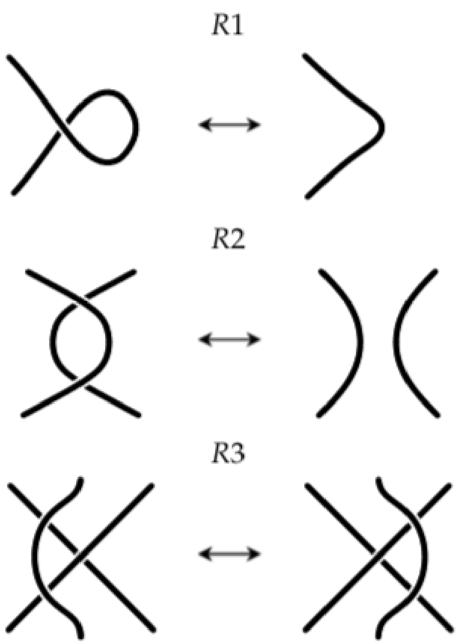


4. Which of these are unknots?



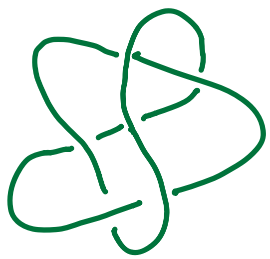
Reidemeister Moves

Definition: A Reidmeister move is one of the following three moves:



Each move has several variations. For example, in a type R3 move, the strand might be entirely under instead of over the crossing.

5. Use a sequence of Reidmeister moves to get from this knot diagram to the standard diagram for the unknot.

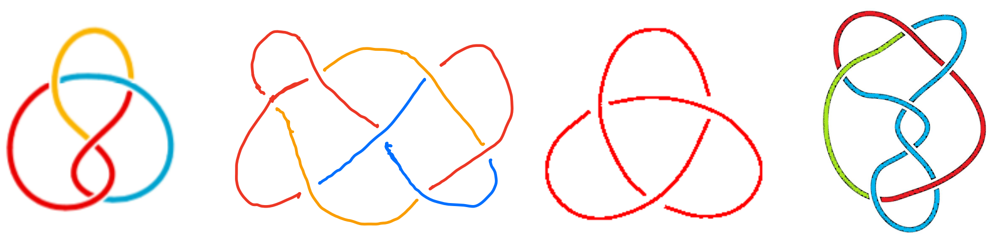


6. Suppose we can get from one knot diagram to another by a sequence of Reidmeister moves. Are the two knots represented by the two knot diagrams necessarily equivalent?
7. Suppose we have two knots that are equivalent, but are represented by two different knot diagrams. Can we get from one knot diagram to the other by a sequence of Reidmeister moves?

Knot Coloring

8. The following knot colorings are legit ”tricolorings”.

The following knot colorings are not legit tricolorings.



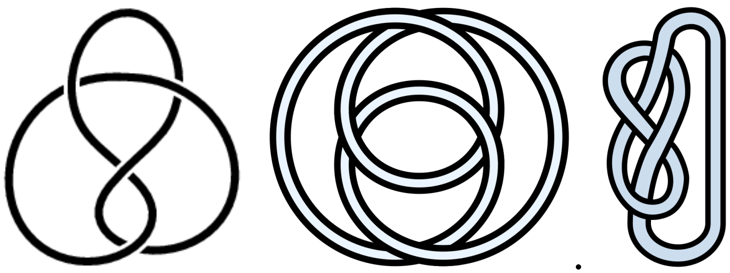
What are the rules are for a knot coloring to be a legit tricoloring?

9. Suppose you and your friend have two different knot diagrams for the same knot. If your friend can tricolor hers, does that necessarily mean that you can tricolor yours?

   In case you want to experiment, here are some different knot diagrams for the trefoil knot.

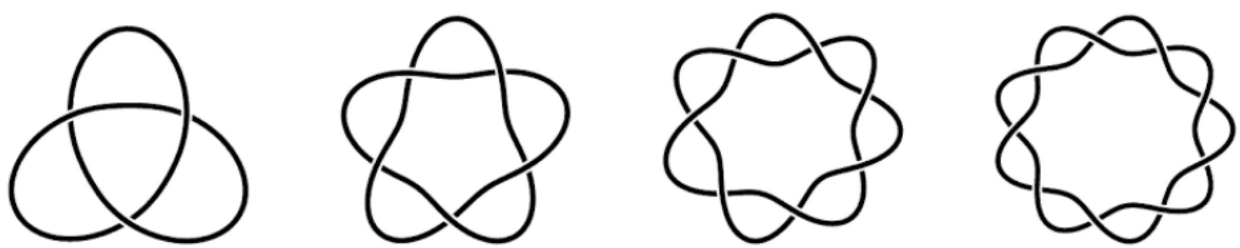
(a) (b) (c)

And here are some different knot diagrams for the figure eight knot.



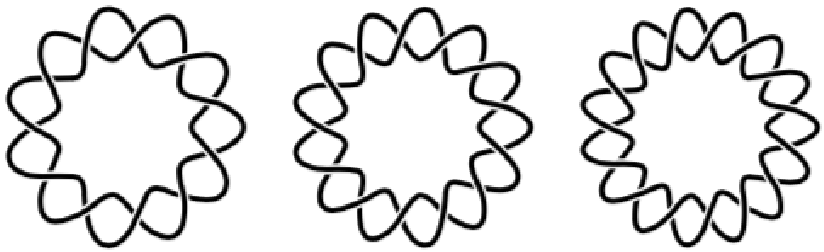
10. Prove that tricolorability is a "knot invariant" – that is, it remains unchanged under Reidemeister moves: if one knot diagram can be tricolored, then another knot diagram of the same knot that differs by a Reidemeister move can also be tricolored.

11. Which of these knots are tricolorable? They are the trefoil, the 5_1 knot, the 7_1 knot, and the 9_1 knot (why do you think they have those names?). What do you notice about your answers? Make a conjecture and explain your reasoning.

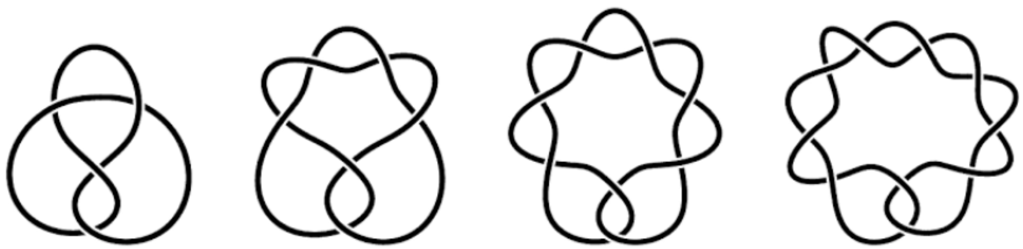


    Note that to prove that a knot is tricolorable, you just need to demonstrate a tricoloring. To prove that a knot is NOT tricolorable, you have to come up with a logical argument to show that no tricoloring at all could be possible.

12. Here are the 11_1, 13_1, and 15_1 knots.



13. Here are the 4_1, 6_1, 8_1, and 10_1. Which are tricolorable? Make a conjecture and explain your reasoning.



Draw the 12_1 knot and check your conjecture.

14. What does tricolorability tell you about the figure 8 knot, the trefoil, and the unknot?

Counting Tricolorings

15. We can also think about not only whether or not a knot is tricolorable, but how many possible tricolorings there are. Let's write $\tau(K)$ for the number of tricolorings of a knot K.

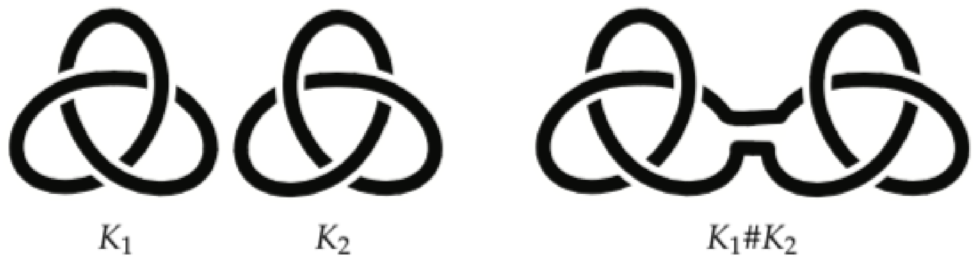
16. Prove that $\tau(K)$ is a knot invariant (i.e. it does not change when you perform Reidemester moves).

17. Compute $\tau(K)$ for the trefoil, the figure 8 knot, and the square knot (shown below as the connect sum of the trefoil and its mirror image). Conclude that these are all different knots! Do you notice a pattern? Can you explain why you see that pattern?

18. The previous problem shows that τ is a more refined invariant than the simple yes-no of tricolorability: it can distinguish between the trefoil and the square knot even though both are tricolorable.

Colorings and Connected Sums

The connected sum $K_1 \# K_2$ of two knots K_1 and K_2 is formed by erasing a little piece of each knot and then connecting the loose strands together, as shown in an example below. This example takes a copy of the right-handed trefoil and the left-handed trefoil and forms their connected sum, which goes by the name of the square knot.



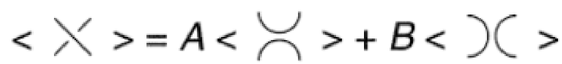
If K_1 and K_2 are tricolorable, is $K_1\#K_2$ tricolorable?

19. Find a formula that relates $\tau(K_1)$, $\tau(K_2)$, and $\tau(K_1\#K_2)$.

Question. Can the unknot be written as the connected sum of two knots?

Knot polynomials

Goal: define an invariant of knots, denoted by bracket symbols <>, that can be computed via a "skein relation":



Question. What does this mean?

We also want the following properties:

$< \bigcirc > = 1$

and

$< L \cup \bigcirc > = C < L >$

We want the invariant to be invariant under Reidemeister moves. Let's see what conditions this forces on A, B, and C.

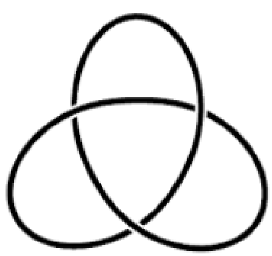
Reidemeister move 1 is problematic:

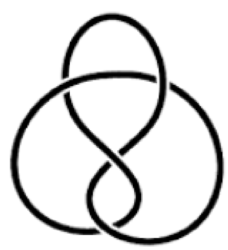
We can salvage the bracket polynomial <> to make it invariant under Reidemeister move 1 (and still invariant under moves 2 and 3) by considering *writhe*.

Definition. The *writhe* of a knot or link is defined as follows:

Assign +1 Assign −1

Example. Compute the writhe of the trefoil knot and the figure 8 knot. Does the orientation that you choose affect your answer?





Definition. Define the Kauffman polynomial for a link L from the bracket polynomial by $K_L(A) =$

Question. Is the Kauffman polynomial invariant under Reidemeister move 1?

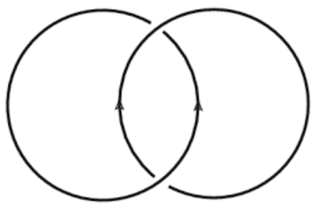
L'

L

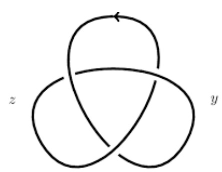
Question. Is the Kauffman polynomial invariant under Reidemeister moves 2 and 3?

Compute $k(L)$ for

1. the unlink with two components. Does it matter how you orient the components?
2. the oriented Hopf Link. What happens if you orient it differently? How. many different orientations are possible?



3. the right-handed trefoil knot. What happens if you orient it the other way?



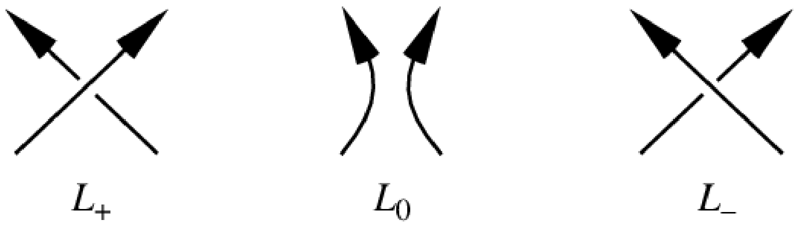
What happens if we switch the right handed trefoil to a left handed trefoil?

The Jones polynomial $V_L(t)$ is obtained from the Kauffman polynomial $K_L(A)$ by replacing each A by $t^{-1/2}$.

Exercise. Write down the Jones polynomial for the right-handed and left-handed trefoil knots.

Exercise. Use the skein relation of the kauffman polynomial to show that the Jones polynomials of the three links below are related through the equation:

$t^{-1}V_{L^+}(t) - tV_{L_-}(t) + (t^{-1/2} - t^{1/2})V_{L_0}(t) = 0$

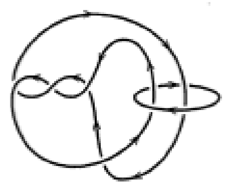


This was the original skein relation that Vaughn Jones recognized to hold for the Jones polynomial.

Question. Does the Jones polynomial always distinguish a knot from the unknot? That is, are there any non-trivial knots with Jones polynomial equal to 1?

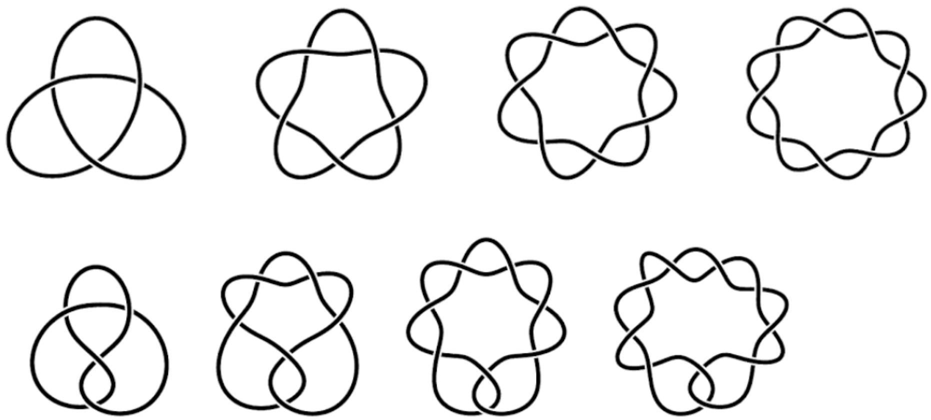
Practice Problems

1. Determine the writhe of this link projection.



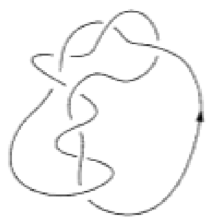
2. Find the Jones polynomial of
   (a) the figure 8 knot
   (b) the Whitehead link
   (c) the Borromean rings
   (d) the square knot

3. Find the Jones polynomials of the unlink with n components, the linear chain with n components, and the knots n_1 below for general n.



4. Let L be a split link consisting of the union of two links L_1 and L_2. Determine how the kauffman polynomial of L is related to the kauffman polynomials of L_1 and L_2.
5. If you have an oriented knot and change the orientation, what happens to the kauffman polynomial?
6. Let L be an oriented link diagram, and let L' be its mirror image. How is $k(L')$ related to $k(L)$?
7. If a knot is equivalent to its mirror image, then it is called achiral (or amphichiral). If it is not equivalent to its mirror image, it is called chiral. Is the trefoil chiral or achiral? What about the figure 8 knot?

8. If a knot is achiral, what can you say about its Kauffman polynomial? Verify that the Kauffman polynomial for the figure 8 knot has this property.
9. Show that this knot below is chiral, not achiral.

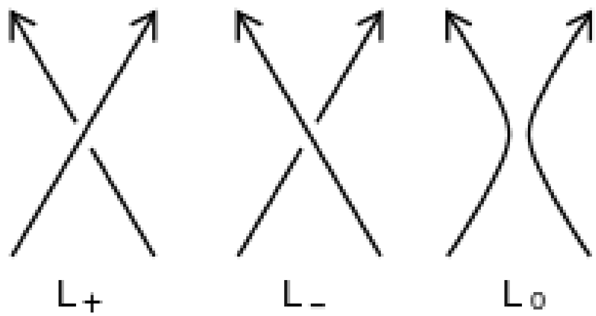


10. Here are a few properties of the Jones polynomial. Verify them for one or two links, and see if you can prove them!
    (a) $V_L(1) = (-2)^{p-1}$, where p is the number of components of the link L.
    (b) If p is the number of components of L, then if p is odd, $V_L(t)$ is a polynomial with integer powers, and if p is even, then $V_L(t)$ is $t^{1/2}$ times such a polynomial.
    (c) $V_{L_1 \# L_2}(t) = V_{L_1}(t) \cdot V_{L_2}(t)$

11. The Alexander polynomial of a link L, $\Delta_L(x)$ is given by the rules:

$$\nabla(O) = 1$$

$$\nabla(L_+) - \nabla(L_-) = z\nabla(L_0)$$



Sometimes the second rule is written

$$\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\Delta(L_0)$$

by replacing the variable z by $t^{1/2} - t^{-1/2}$.

Although we will not prove it, the Alexander polynomial is an invariant of oriented knots and links.

(a) Compute the Alexander polynomial of a right-handed trefoil and a left-handed trefoil.

(b) Compute the Alexander polynomial of a figure 8 knot.

(c) Show that the Alexander polynomial of a splittable link is always 0. Hint: picture the splittable link as L_0.

(d) Can you find a pair of knots or links whose Alexander polynomials are the same but whose Kauffman polynomials are different? What about vice versa?

Here is a table of knots which can be represented with only a few crossings.