§5 Euler Characteristic

The goals for this part are

- to calculate the Euler characteristic of surfaces
- to use Euler characteristic to prove certain constructions are impossible
- to use Euler characteristic to identify surfaces.

References:

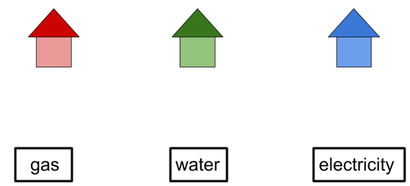
- *The Symmetries of Things* by Conway, Burgiel, and Goodman-Strauss, chapter 7
- Geometry Junkyard's 20 proofs of Euler's Theorem
- *Circle in a Box* by Sam Vandervelde, Chapter 5

Supplies:

- Snap together polygons
- Soccer ball
- Zome
- Balloons and sharpies or whiteboard markers
- Criss-cross game boards

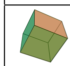
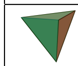
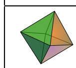
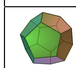
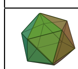
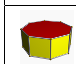
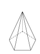
The gas, electricity, and water problem

Suppose there are three cottages on a plane and each needs to be connected to the gas, water, and electricity companies. Using a third dimension or sending any of the connections through another company or cottage is disallowed. Is there a way to make all nine connections without any of the lines crossing each other?



Faces, vertices, and edges of polyhedra

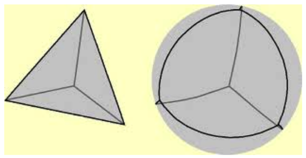
A polyhedron is a 3-dimensional shape with flat polygon faces, straight edges, and sharp corners, called vertices. For example, a cube and a tetrahedron are polyhedra. For each of these polyhedra, count the faces, edges, and vertices.

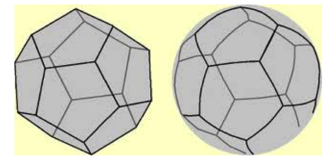
Object	Faces (F)	Edges (E)	Vertices (V)
 Cube			
 Tetrahedron			
 Octahedron			
 Dodecahedron			
 Icosohedron			
 Prism on n-sided base			
Pyramid on n-sided base 			
Pentagonal Cupola 			
Soccer Ball 			

- Find a formula relating the number of faces, edges, and vertices of a polyhedra.
- This formula is known as *E*uler's formula.

Euler's fomula on the sphere and on the plane

- Note: We don't really need the faces to be flat or the edges to be straight to count them. We could imagine blowing air into a polyhedron and puffing it up like a balloon and we could still count faces, edges, and vertices on the balloon surface.

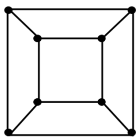




- A network of vertices, edges, and faces on the sphere is called a *m*ap on the sphere.
- Try to break Euler's formula: draw a map on the sphere for which it does not hold.

- What conditions do we need on the faces and the edges to make sure that Euler's formula always holds for any map on a sphere?

Note. One way to represent a polyhedron on a flat piece of paper is as follows. Imagine the polyhedra is drawn on a flexible rubber ball. Take any face and punch a hole in it, then stretch the edges of that hole until the hole is much bigger than the original polyhedra. For example, this would turn a cube into the following figure. The punctured face is now the infinite outside region of the figure.



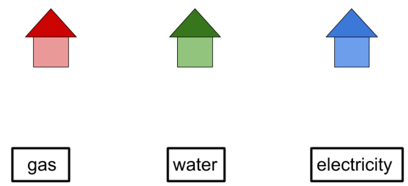
Exercise. Represent a tetrahedron and your other favorite polyhedra in a similar way. We won't worry about straight edges anymore.

Definition. A *planar graph* is a collection of points, called vertices, and line segments, called edges, drawn on the plane, such that each edge connects two vertices (which might both be the same vertex) and edges only meet at vertices (they don't cross each other).
Exercise. Draw a few planar graphs.

Question. Does Euler's formula hold for planar graphs?

Gas, water, and electricity, revisited

What does Euler characteristic have to do with the gas, water, and electricity problem?

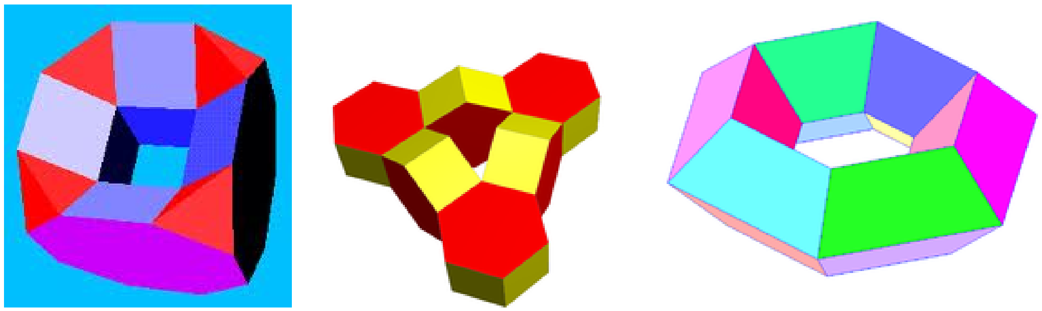


Example. Find a way to position four points on a sheet of paper so that when every pair of points is joined by a straight line segment, none of the segments intersect. (How many segments will there be?)

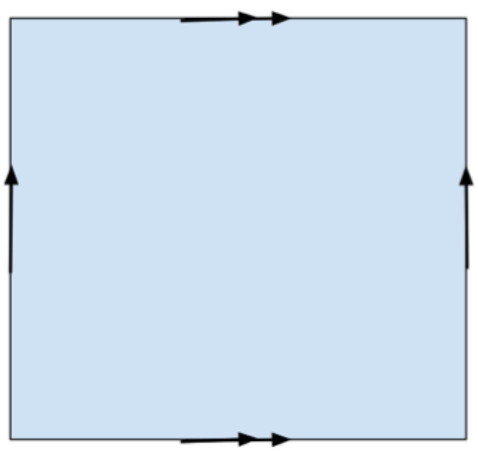
Example. Is it possible to draw 5 points on the plane and connect each pair of points with a straight line segment in such a way that the segments do not cross? Prove your answer.

Euler characteristic for other surfaces

Question. Does Euler's formula still hold for the vertices, edges, and faces of a polyhedral torus?



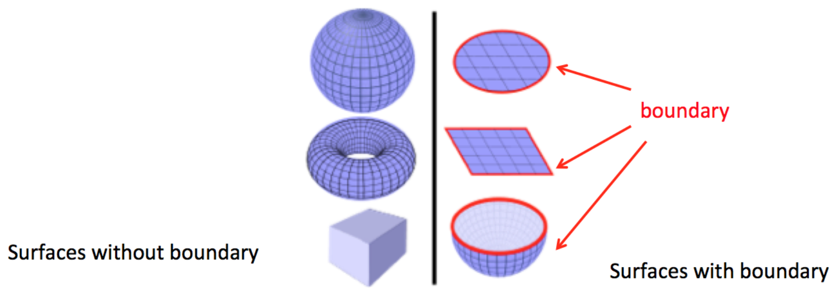
Question. How could you compute the Euler characteristic of the torus just from a gluing diagram?



Question. Is it possible to solve the gas, water, and electricity problem on the torus?

Euler characteristic for surfaces with boundary

Question. Is it possible to compute Euler characteristic for surfaces with boundary?



Find the Euler characteristic of:

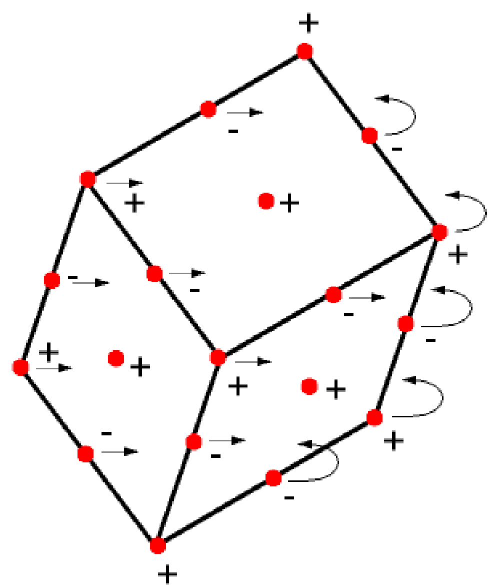
- A disk

- A cylinder

- A Mobius band

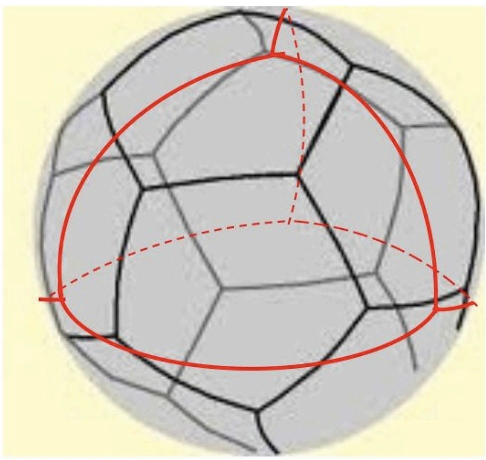
An electric charge proof of Euler's formula for the sphere.

Question. Why does Euler's formula hold on the sphere?



An inking-in proof of Euler's formula

$V - E + F$ using the red lines is the same as $V - E + F$ using the black lines.



- The inking-in proof works for any topological surface, not just a sphere, to show that $V - E + F$ is the same for any map on that surface.
- Any two polyhedra that have the same topology have the same Euler characteristic.
- The Euler characteristic is called a topological *invariant* of the surface.

Additional proofs of Euler's formula for the sphere can be found at David Eppstein's Geometry Junkyard website. Look for Twenty Proofs of Euler's Formula $V - E + F = 2$

Problems on Euler Characteristic

1. In a certain small country, there are villages, expressways, and fields. Expressways only lead from one village to another and do not cross one another, and it is possible to travel from any village to any other village along the expressways. Each field is completely enclosed by expressways and villages. If there are 70 villages and 100 expressways, how many fields are there?
2. Is it possible to solve the houses and utilities problem on a torus? Demonstrate a solution or show that it is not possible.
3. Find the Euler characteristic of the following surfaces:
   (a) the projective plane (Hint: work from the gluing diagram.)
   (b) the Klein bottle
   (c) a pair of pants
4. A polyhedron is made up of pentagons and hexagons in such a way that three polygons meet at each vertex. How many pentagons must there be? Prove that no other number of pentagons is possible. Challenge: what are the possible answers for the number of hexagons?
5. A certain polyhedron is built entirely from triangles, in such a way that 5 faces meet at each vertex. Prove that it has to have 20 faces. (Hint: first deduce that 3F

= 2E and 3F = 5V)

6. Another polyhedron is built entirely from triangles, in such a way that 4 faces meet at each vertex. Prove that it has to have 8 faces. What about if 3 faces meet at each vertex? What happens if 6 faces meet at each vertex?
7. A polyhedron is called regular if all its faces are the same regular polygons (for example, all equilateral triangles or all squares) and if the same number of faces meet at each vertex (for example, 3 faces meet at each vertex). For example, the cube, the tetrahedron, the octahedron, the dodecahedron, and the icosahedron are all regular polyhedra, but the shape made by gluing two tetrahedra together along a triangle is not, because some vertices have 4 triangles around them and others have 3. Prove that the five regular polyhedra mentioned in the previous sentence are the only regular polyhedra possible. (Hint: start your argument by using the problems above.) Regular polyhedra are also called Platonic solids.
8. Using Euler's Formula, prove that you can't connect eight points pairwise on a torus, in such a way that none of the connecting curves intersect. ("The complete graph K8 cannot be embedded in a torus.")
9. Connect five points pairwise on a torus, in such a way that none of the connecting curves intersect. Now do six points; then seven (hard!).
10. What's the maximum number of points that can be connected pairwise on the projective plane? On a Klein bottle?

11. Prove that any triangulation of the torus must have at least 7 vertices. That is, your solution to the previous problem (connecting 7 vertices pairwise on the torus) constitutes a minimal triangulation of the torus. What's the analogous result for the projective plane?
12. A certain polyhedron is made of squares, hexagons, and decagons (which are 10-sided polygons), in such a way that one square, one hexagon, and one decagon meet at each vertex. How many vertices does it have?
13. What are 4-dimensional analogs of cubes and other polyhedra? How could you define Euler characteristic for them? What number do the parts sum to?
14. What patterns hold for the number of faces, edges, and vertices for these planar graphs (also known as *scribbles*)? (A planar graph is a collection of vertices and edges drawn in the plane, in such a way that there is a vertex at the beginning and end of each edge and at any place where two edges meet or cross.) Note that the planar graph (or scribble) can have more than one component.

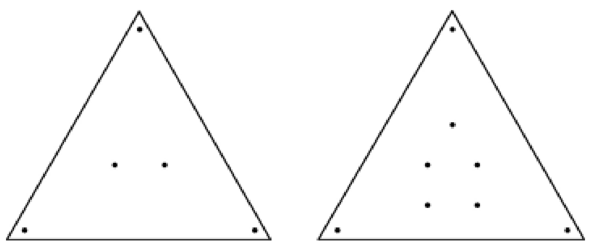
15. Prove the formula you found for scribbles, which should be a generalization of

Euler's formula. Hint: first, show that the formula holds for a scribble with 0 edges. Then use an induction argument based on the number of edges. What happens to the pattern you have observed if you remove an edge from a scribble?

Many of these problems are from *Circle in a Box* by Sam Vandervelde and from Beginning Topology by Sue Goodman.

The Game of Criss-Cross

The game of Criss-Cross is played on a gameboard created by drawing three points at the vertices of a large equilateral triangle, along with two to seven additional points anywhere in its interior. Players alternate turns drawing a single straight line segment joining any two points, as long as the segment does not pass through any other points or segments already appearing on the game board. The winner is the last player able to make a legal move.



Two sample Criss-Cross boards

Exercise. Play some games of criss-cross with a classmate. Is there a winning strategy? Do you want to go first or second? Analyze the game!

Euler characteristic of some familiar surfaces

Find the Euler characteristic for:

1. a torus
2. a 2-holed torus
3. a cylinder (without the top or bottom)
4. a cone (without the bottom)
5. a hexagon
6. a Mobius band

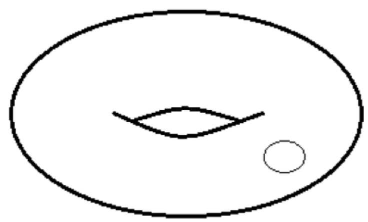
Note that the cylinder, cone, hexagon, and Mobius band are *s*urfaces with boundary.

Are all integers possible as the Euler number for surfaces?

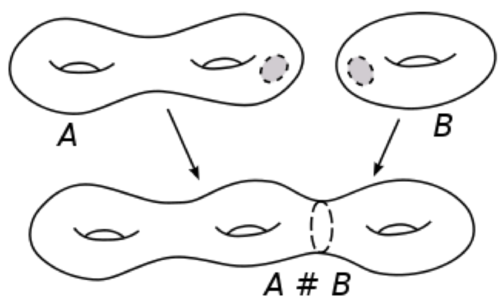
Building new surfaces out of old

Two ways of building new surfaces (with and without boundary) out of old are:

1. making punctures



2. joining surfaces with connected sums



Question. What do punctures do to Euler characteristic?

Question. What do connected sums do to Euler characteristic?

What Euler characteristics are possible?

For surfaces without boundary?

$$S^2$$

T^2	P^2
$T^2\#T^2$	$P^2\#P^2$
$T^2\#T^2\#T^2$	$P^2\#P^2\#P^2$
$T^2\#T^2\#T^2\#T^2$	$P^2\#P^2\#P^2\#P^2$
$\dots$	$\dots$

For surfaces with boundary?

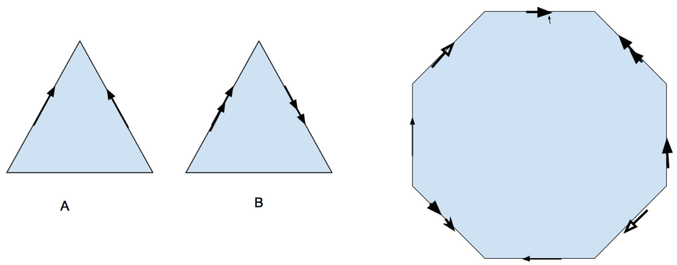
Classify surfaces by Euler characteristic (and orientabililty)

What surfaces have

- Euler characteristic of 2?
- Euler characteristic of 1?
- Euler characteristic of 0?
- Euler characteristic of -1?

What surface is this?

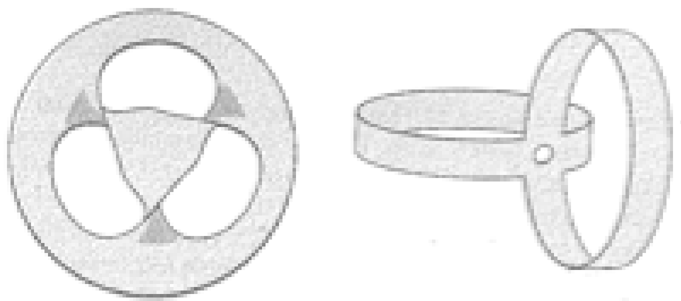
Exercise. What topological surfaces do each of these figures represent? Use Euler characteristic, orientabiity, and number of boundary circles to decide.



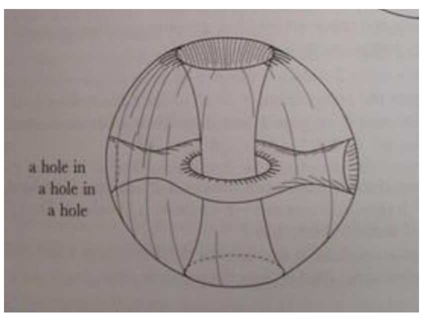
Single elimination gluing diagram identification tournament. See who can correctly identify the surface from the gluing diagram first. The loser gets to draw the gluing diagram for the next pair of contestants.

Problems on identifying surfaces using Euler characteristic

1. Identify these surfaces.



2. Identify this surface that is a hole within a hole within a hole.



Hole in a hole in a hole

See also the Numberphile video.

3. Take a piece of paper in the shape of a cross, and identify opposite ends The result is a surface with boundary. How many boundary components are there? If you were to describe the result as a closed surface M with n punctures, what would M

be and how many punctures? If your answer depends on the gluing instructions for the ends be sure to cover all cases.

4. Identify the surfaces obtained by
   (a) Gluing the opposite sides of an octagon as an orientable surface.
   (b) Gluing the opposite sides of a decagon as an orientable surface.
   (c) Gluing the sides of a polygon with gluing instructions given by the word $b_1a_2b_2a_3^{-1}b_2^{-1}a_3a_2^{-1}a_1^{-1}b_1^{-1}a_1$
   (d) Gluing the sides of a polygon with gluing instructions given by the word $b_1^{-1}a_2b_2a_3^{-1}b_2^{-1}a_3a_2^{-1}a_1^{-1}b_1^{-1}a_1$
   (e) Gluing two polygons with words around the outside of: $ad^{-1}e^{-1}fga^{-1}b^{-1}c^{-1}$ and $debf^{-1}cg$
5. A hexagon has its edges identified in pairs of letters a, b, c to give a surface. How many distinct (not topologically equivalent) surfaces can be represented in this way? Give a word representing each such surface. Hint: Begin by finding upper and lower bounds for the Euler characteristic, thus reducing to a small number of cases for which you need to construct an example.
6. We have seen that $M\#S^2 \equiv M$ for any surface M with pictures. Prove this same fact using Euler characteristic.