§6 Curvature

The goal for this part is to define and quantify curvature in several different ways and to calculate the curvature of surfaces.

References:

- *Geometry and the Imagination* by John Conway, Peter Doyle, Jane Gilman, and Bill Thurston.

Supplies:

- bananas, oranges, lettuce or kale, cabbage
- protractors
- blue tape
- scissors and tape

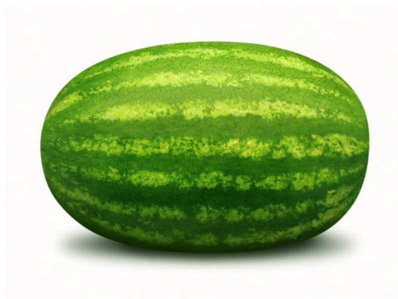
Swimming pool analogy

Suppose you have a swimming pool with an oddly shaped, slanted bottom and you want to talk about the amount of water in the pool. If you are interested in the amount of water below a specific point on the surface of the pool, you will be asking about the depth of the pool at that point, which is measured in units like feet or meters. If, instead, you want to know the amount of water below a small region of the surface of the pool, you will be asking about volume, measured in units like cubic feet or cubic meters. This volume can be approximated as the depth below some point in the small region times the area of the region. Finally, if you want the water in the entire pool, you again will be asking about volume. If you can't calculate it conveniently by draining the entire pool, you could estimate it by dividing the surface into many small regions, approximating the volume below each small region, and adding these volumes up. Students who are familiar with calculus may recognize that what we are really doing here is integrating: we integrate the depth of the pool over the surface of the pool.

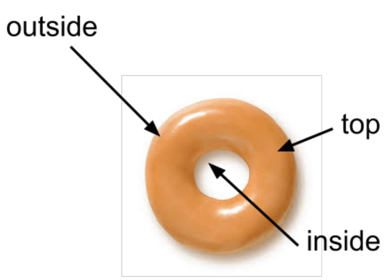
When we talk about curvature of a surface, there are also three ways to describe it, which I will call pointwise curvature, regional curvature, and total curvature. Pointwise curvature refers to the curvature at a point of the surface, analogous to the depth of water below a single point. Regional curvature refers to the curvature within a small region of the surface, analogous to the volume of water below a region of the swimming pool. We will see that regional curvature and pointwise curvature have different units, just like depth and volume. Finally, total curvature refers to the curvature of the entire surface, which, analogous to the swimming pool, can be found by adding up regional curvatures or by integrating pointwise curvature over the entire surface.

Intuition about curvature

Which would you say is more curved, a piece of the surface of an orange or the same size piece of the surface of a watermelon?

Examine the surface of a Krispy Kreme donut.

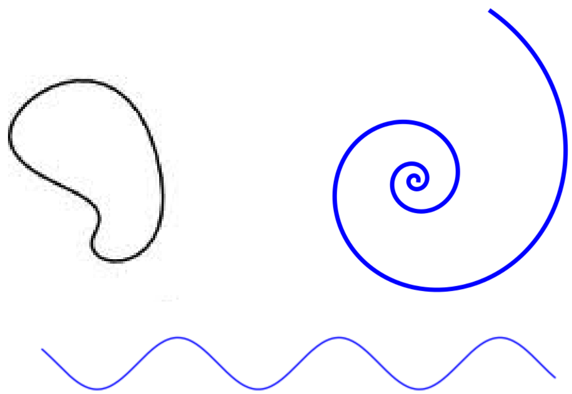


How does the curvature on the inside of the hole compare to the curvature on the outside or on the top?

Curvature of Curves in the Plane

Before we define curvature of a surface, let's drop down a dimension and think about curvature of 1-dimensional curves.

For each of the three curves drawn below, describe the curvature at various points along the curve. For each curve, where is the curvature biggest and where is it smallest?

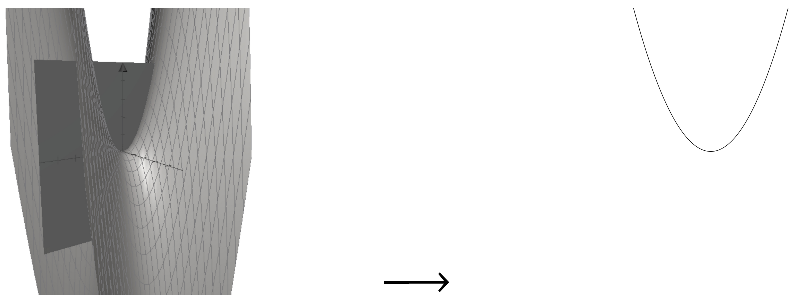


Draw a curve that has the same curvature at all points. How could you quantify its curvature as a number?

Extend this idea to define the curvature at points of other curves, like the spiral or the sine curve (the wave above).

Gaussian Curvature

Once we have a notion of curvature of curves in a plane, we can use this to define the curvature at a point on a surface in various directions. Take a point on the surface, and imagine slicing the surface by a plane perpendicular to the surface at that point. The intersection of the 2-dimensional surface and the plane makes a 1-dimensional curve in the plane. This 1-dimensional curve has a curvature that we'll call a *slice* curvature of the surface.



At any point, there could be infinitely many slice curvatures, depending on which direction you slice the surface with a plane. If the surface bends in opposite directions, as happens at a saddle point, then we consider some slice curvatures positive and others negative.

At any point of the surface, the maximum and minimum slice curvatures are called the *principal curvatures* at that point. It is a fact (that we won't prove) that the principal curvatures always lie in perpendicular directions.

One way to get a single number for the curvature of a point on the surface is to multiply together the principal curvatures. This product is called the *Gaussian curvature* at that point on the surface.

Exercise. According to this definition, is the curvature at a point on a watermelon bigger or smaller than the curvature at a point on an orange?

Exercise. Estimate the curvature at various points on a bagel. Where is the curvature greatest and least? Where is the curvature positive, negative, and zero?

Exercise. What is the curvature of a point on a cylinder?

Exercise. What is the curvature of a point on the side of a cone?

Exercise. What are the units of curvature at a point?

Exercise. Calculate the curvature at each point of a sphere of radius 9 cm.

Exercise. Calculate the regional curvature of the upper hemisphere of a sphere of radius 9 cm. Since the curvature at each point is the same, you can find the regional curvature by multiplying the curvature at a point by the area of the region.

Exercise. What are the units of regional curvature?

Flattening curved surfaces

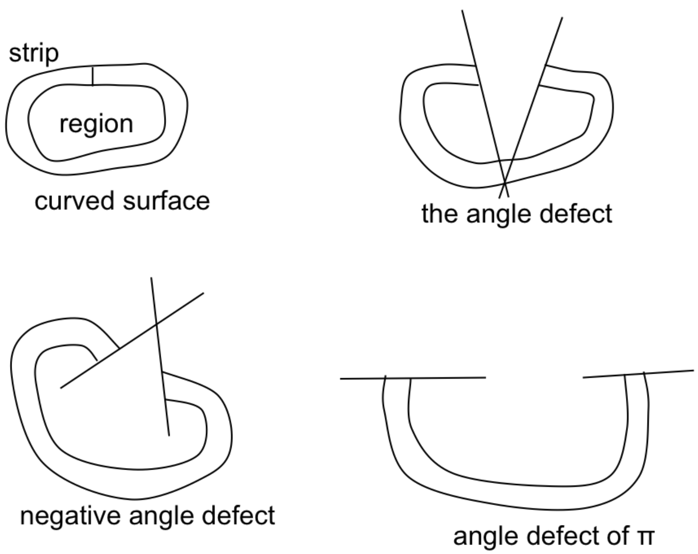
One way you know that a surface is curved is that it doesn't flatten without tearing or stretching.

Question. How can you quantify the curvature in a piece of orange peel?

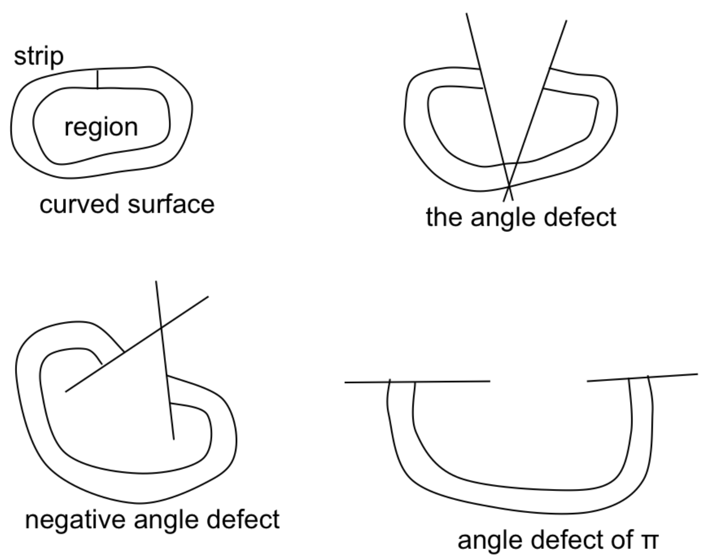
Angle defect

One way to quantify the amount of curvature inside a region of surface is by calculating angle defect.

1. First, cut out a disk-like region of the surface.
2. Next, cut a thin strip around the boundary of this region.
3. Fatten the strip out as much as possible.
4. Measure the angle between the edges of the strip. This is the curvature inside the region.



If the strip meets up with itself perfectly, then the region has zero angle defect regional curvature. Sometimes, the strip doesn't meet up because it doesnÕt curl enough. This is positive curvature. Sometimes the strip doesn't meet up because it curls around too much and overshoots. This is negative curvature.



If we measure angle defect in radians instead of degrees, then the angle defect definition of regional curvature will agree with the previous definition based on circles. Note also that the region must have the "topology of a disk" for the angle defect definition to work. In other words, the region should not contain any holes or handles. So a small piece of the surface of a bagel is fine, but a large piece that contains the entire hole in the middle is not okay.

Find the regional curvature of:

a piece of orange peel

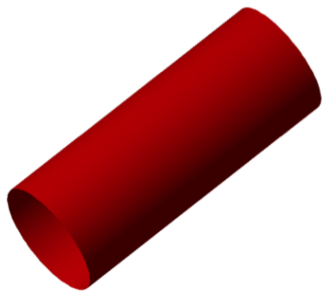
a piece of a kale leaf

a piece of banana peel

a piece of a cabbage leaf

One nice feature about the angle defect regional curvature is that it is possible to measure curvature even at regions of a surface that are not smooth, like the cone points on a cone or vertices on a polyhedron.

Find the angle defect of



a piece of the side of a cylinder

a piece of a cone

that does not contain the cone point

Which cone has more angle defect around its cone point?