§7 The Gauss-Bonnet Theorem

The goal for this part is to state and prove a version of the Gauss-Bonnet Theorem, also known as Descartes Angle Defect Formula. This theorem relates curvature (geometry) to Euler characteristic (topology).

References:

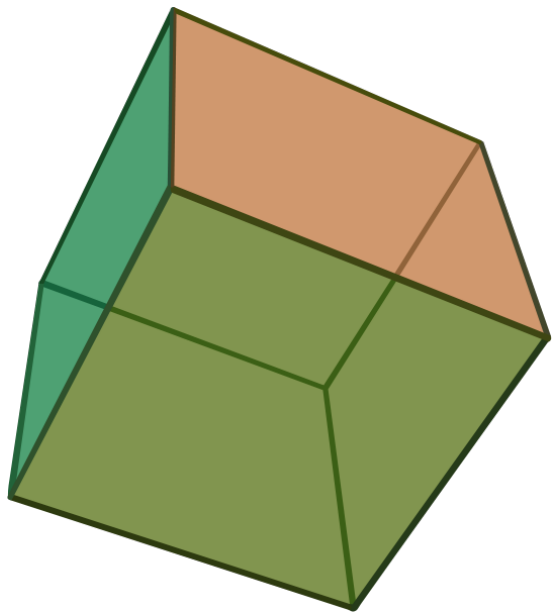
- *Geometry and the Imagination* by John Conway, Peter Doyle, Jane Gilman, and Bill Thurston.

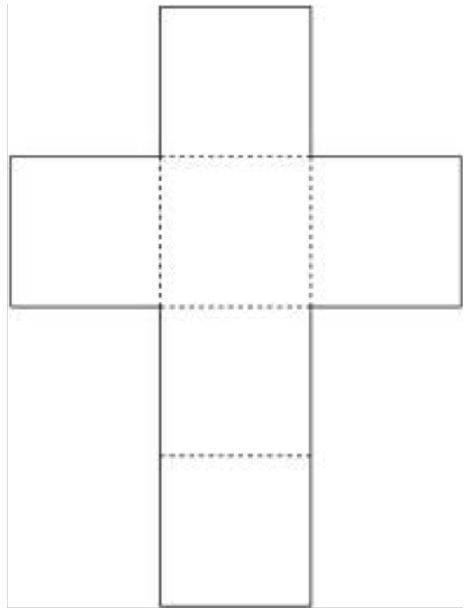
Supplies:

- polyhedra
- equilateral triangle paper

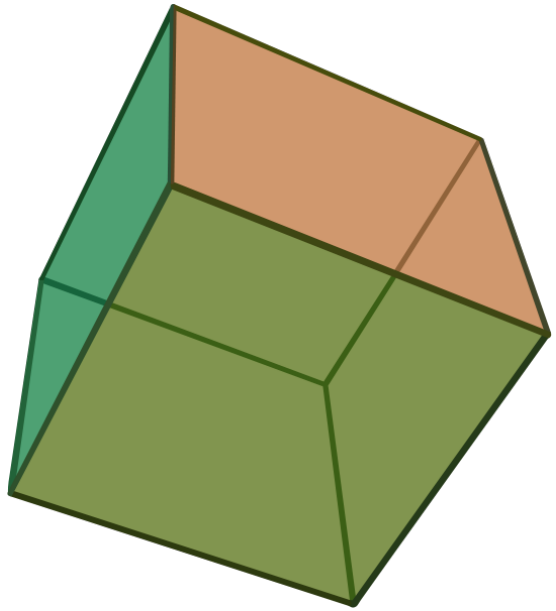
Total Angle Defect ⟷ Total Curvature

Find the angle defect of a piece of the surface of a cube that contains one vertex.





You can find the total curvature (total angle defect) of a surface by dividing it into many small regions and adding up the curvature (angle defect) of each region.



What is the total curvature of a cube?

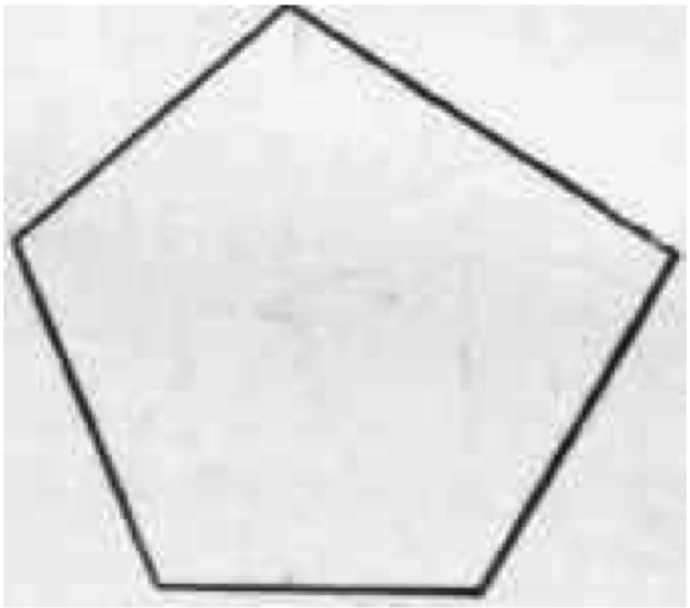
Total curvature of polyhedra

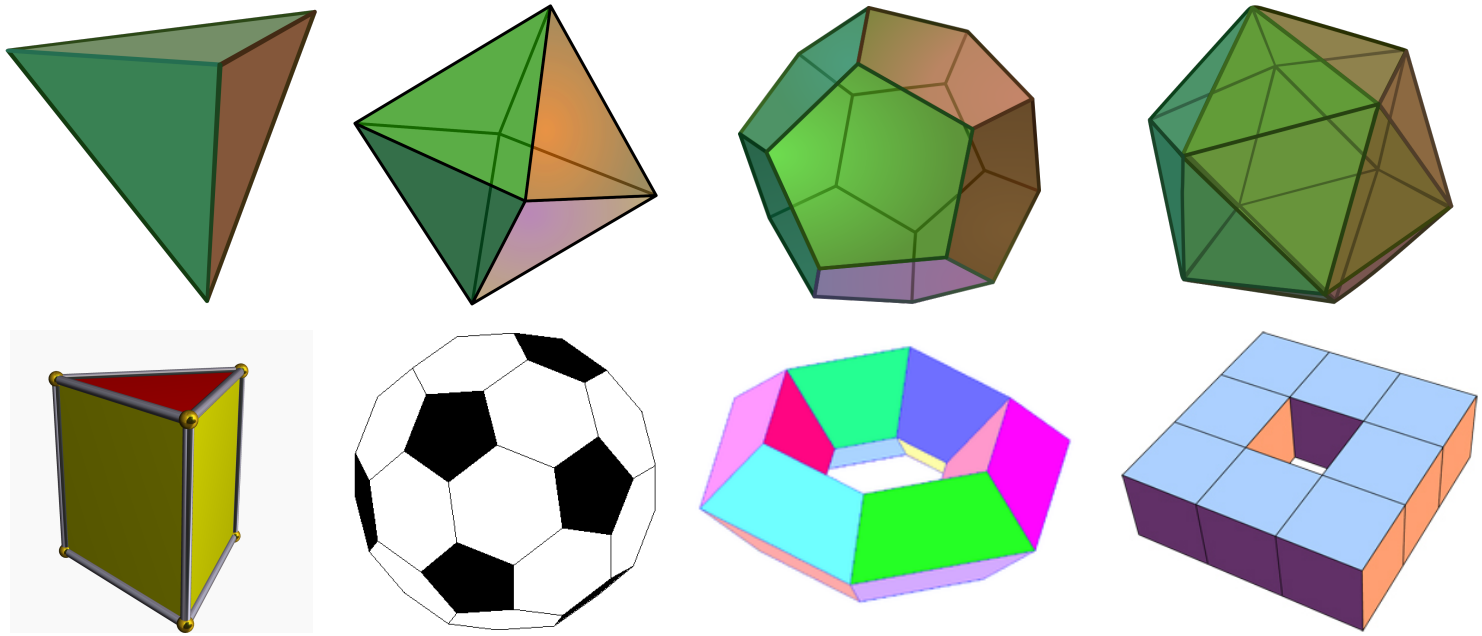
On a polyhedron, the angle defect of any region that doesn't contain a vertex is 0. Why?

We can calculate the total angle defect of a polyhedron by adding up the angle defects at each vertex.

It will be handy to answer this question first:

What is the sum of the interior angles of a polygon with n sides?





What do you notice about the total curvature of these objects?

Descartes Angle Defect Formula

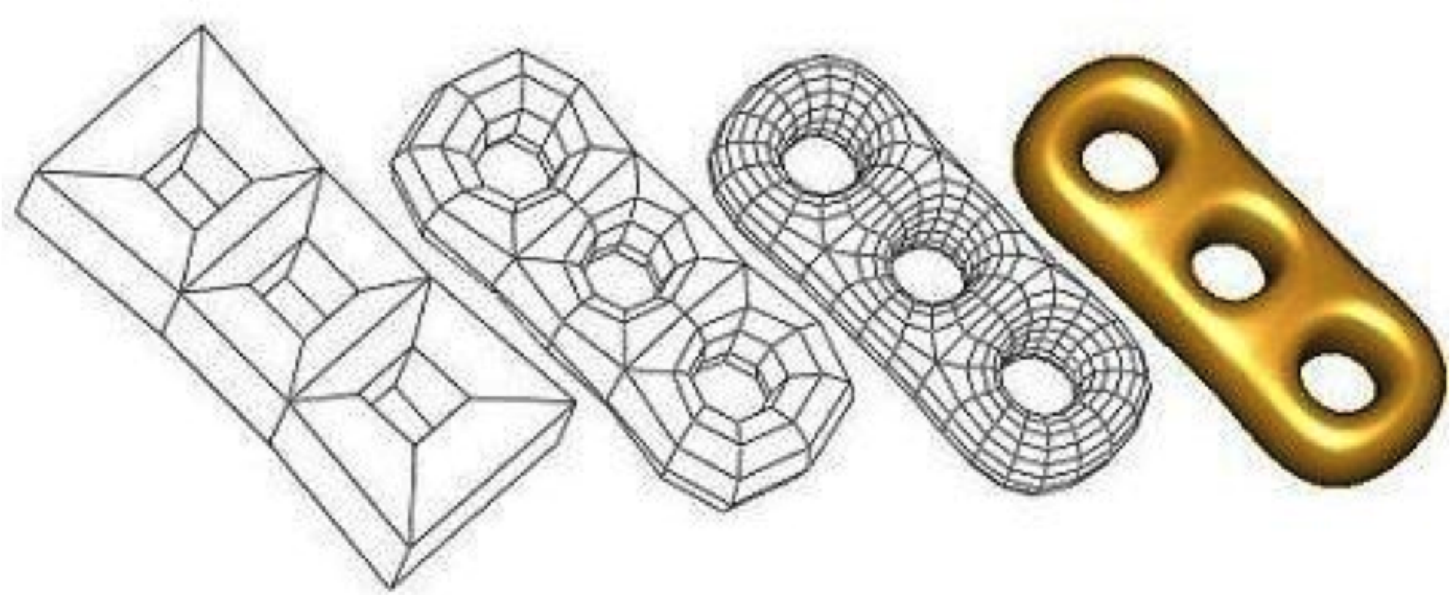
Descartes Angle Defect Formula says:

Electric charge proof of Descartes Angle Defect Formula

Alternate proof of Descartes Angle Defect Formula

Gauss-Bonnet Theorem for Smooth Surfaces

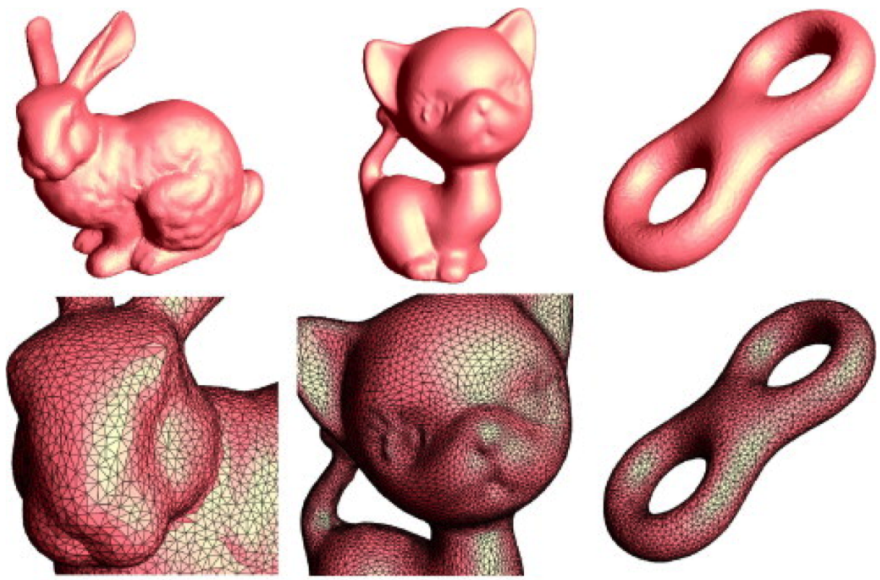
Question. What about smooth surfaces that are not polyhedra?



Reference image: CGAL, Computational Geometry Algorithms Library, http://www.cgal.org

- We can approximate any smooth surface with polyhedra.
- A limiting argument shows that total curvature $\kappa = 2\pi\chi$ for any topological surface.
- This is called the Gauss-Bonnet Theorem.
- The Gauss-Bonnet Theorem and Descartes Angle Defect say essentially the same thing: one is for smooth surfaces and one is for polyhedra.

Polyhedral approximations to surfaces



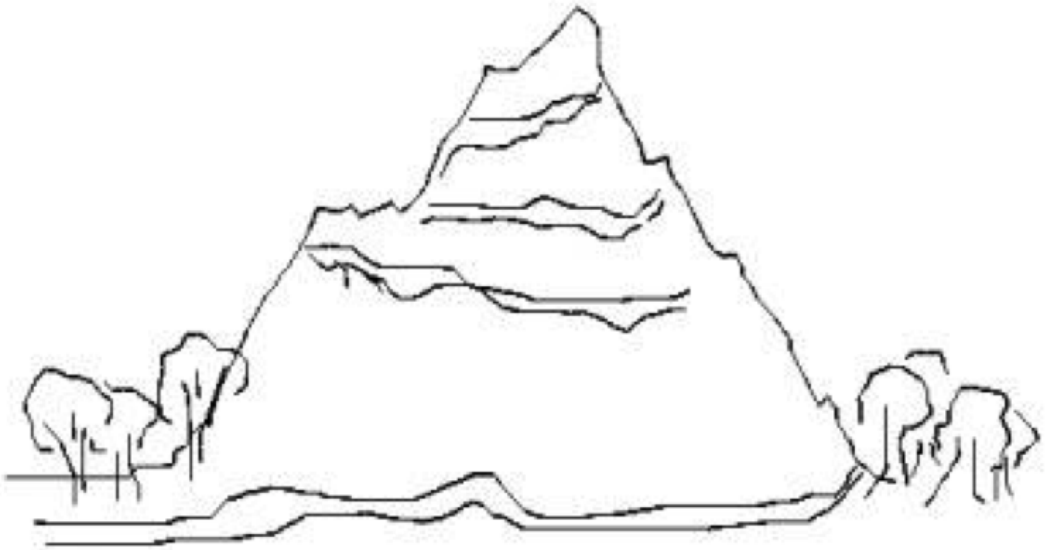
Genus 0 Genus 1 Genus 2

From Wei Zeng, Ren Guo, Feng Luo, Xianfeng Gu, Discrete heat kernel determines discrete Riemannian metric, Graphical Models, Volume 74, Issue 4, July 2012

Why is the Gauss-Bonnet Theorem so amazing?

- Euler number only depends on the *topology* of the surface
- But regional curvature (angle defect) depends very much on the *geometry* of the surface
- Somehow all the ins and outs of regional curvature always exactly balance out so that total curvature ultimately only depends on the topology, not the geometry of the surface

The Gauss-Bonnet Theorem says that for ANY surface, total curvature = $2\pi\chi$.

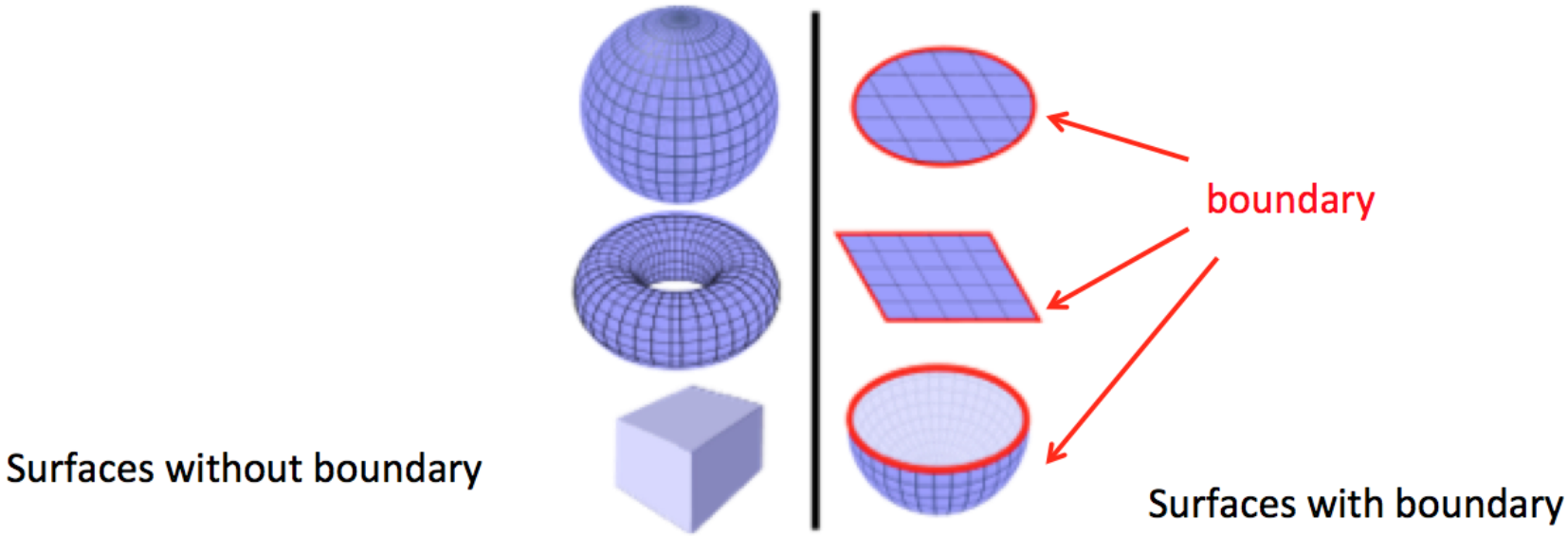


If an earthquake creates a new mountain tomorrow, generating additional positive curvature at the top of the mountain, that new positive curvature has to be exactly balanced by new negative curvature elsewhere.

Where is the negative curvature?

Gauss-Bonnet for surfaces with boundary

So far, we have only considered total curvature for surfaces without boundary

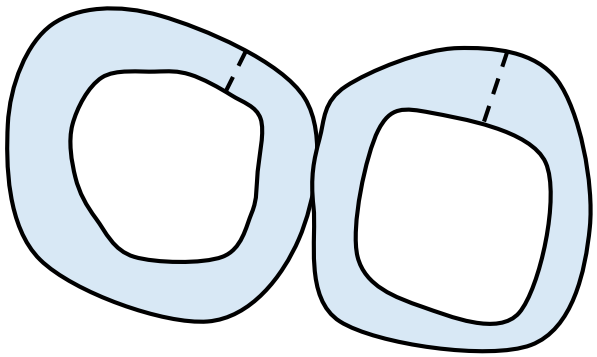


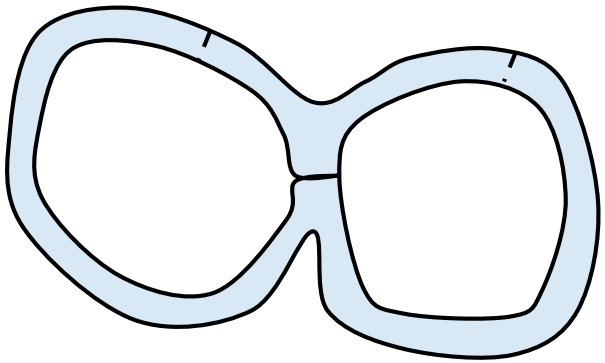
- Is there a Descartes Angle Defect Formula / Gauss-Bonnet Theorem for surfaces with boundary?
- State and prove Descartes Angle Defect Formula for surfaces with boundary
- Be careful about how you define angle defect for vertices on the boundary

Proof of Descartes Angle Defect Formula for Surfaces with Boundary

Problems on Curvature and the Gauss-Bonnet Theorem

1. Verify Descartes Angle Defect Theorem for a surface that is a flat equilateral triangle. By verify, I mean, calculate the Euler number and the total curvature , counting carefully along the boundary vertices, and check that these quantities have the correct predicted relationship.
2. Find the total curvature of the following surfaces:
   (a) A Klein bottle.
   (b) A Mobius band.
   (c) A cylinder, without the top and bottom.
   (d) A disk (that is, a surface that is the inside of a circle).
3. When calculating total curvature, we implicitly used the following fact: If you take two adjacent pieces of a surface, the regional curvature in both pieces put together equals the sum of the regional curvatures in each piece. Why is this true?





4. Construct a surface from equilateral triangles by putting seven triangles around each vertex. What is the curvature of a piece of this surface containing one vertex?
5. Use a strip around the equator to calculate the curvature in the northern hemisphere of a round sphere? The southern hemisphere?
6. What is the curvature on a region of the sphere that is outside of a tiny circle?

Celestial Sphere

Imagine walking around on a surface with a flashlight pointed straight overhead at all times. The flashlight sweeps out a pattern on the "celestial sphere" which you can imagine as a very large sphere surrounding you and the surface.

- What pattern is swept out on the celestial sphere if you travel on the surface of a cube, walking in a loop around each vertex in turn? What if you travel around each vertex of an icosohedron?
- On a convex polyhedron in which three faces meet at each vertex, the celestial image of a path around a vertex is a triangle. Show that the three angles of this celestial triangle are the supplements of the angles of the faces meeting at the vertex.
- Show that the area of this celestial triangle is the angle defect around the vertex. You will need to measure angle defect in radians and you will need to use Girard's formula: area of a triangle on a unit sphere = sum of angles - π. Assume that the radius of the celestial sphere is 1 unit.

  The curvature of a piece of a surface – any surface including smooth surfaces, not just polyhedra – can also be defined as the area inside the celestial image of the piece of the surface. (The piece of the surface should have the topology of a disk for this definition to work.) Let's call this the *Gauss map regional curvature*. Note:

the previous exercise shows that the Gauss map regional curvature is the same as the angle defect regional curvature, at least in the case of a polyhedron with three faces around each vertex.

- What is the celestial image of a spherical triangle from a sphere of radius r? What is its Gauss map regional curvature? What is its area? What is the original Gaussian curvature (defined as the product of principal curvatures) at each point of the sphere? Try to relate the Gauss map regional curvature to the original Gaussian curvature.