§9 Combining isometries

The goal for this lesson is to figure out what happens when we combine isometries; that is, do one isometry followed by another.

Supplies:

- Geogebra (phones, laptops)
- Hinged mirrors and flat mirrors (kaleidoscope mirrors)

Reflections of reflections

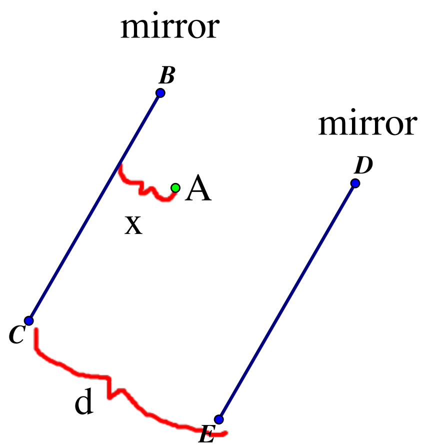
- Suppose we have two parallel mirror lines, and we do a reflection through the first mirror line followed by a reflection through the second mirror line. What is the isometry that results?

- Suppose we have two mirror lines that are NOT parallel (they intersect), and we do a reflection through the first mirror line followed by a reflection through the second mirror line. What is the isometry that results?

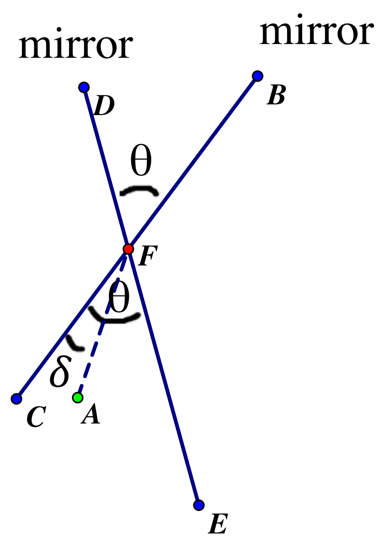
Mirrors are available, or you may want to use Geogebra to help.

Angles and distances for products of two reflections

Reflect the point A through the mirror BC and draw the image point A'. Then reflect A' through mirror DE and draw its image A''. What is the distance between A and A'' in terms of the distances d and x?



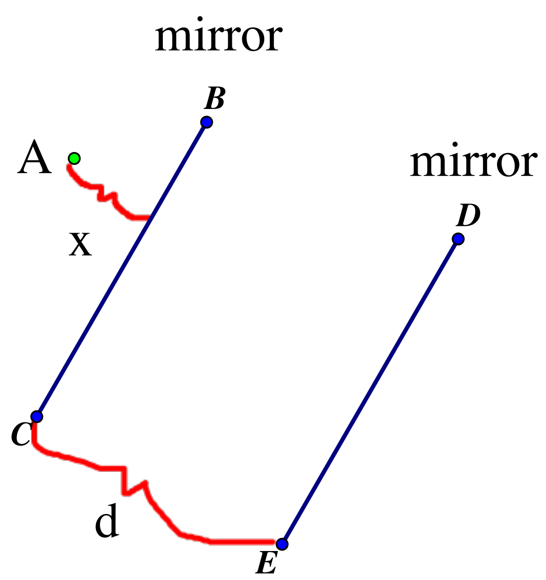
Reflect the point A through the mirror BC and draw the image point A'. Then reflect A' through mirror DE and draw its image A''. What is the angle $\angle ABA''$ in terms of the angles δ and θ?

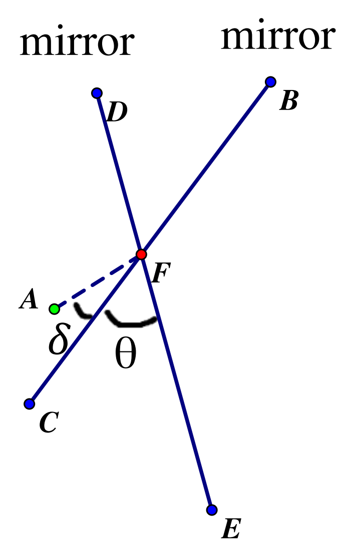


What happens if the reflections are done in the opposite order?

What happens if the point is outside the two mirrors instead of between them?

For each figure, reflect the point A through the mirror BC and draw the image point A'. Then reflect A' through mirror DE and draw its image A''. Describe the resulting isometry (e.g. translation, etc.) precisely: for example, if the answer is a translation, give the direction and length of the translation vector, etc. Give your answers in terms of d and x or δ and θ and the positions of the original mirrors. Explain your work.





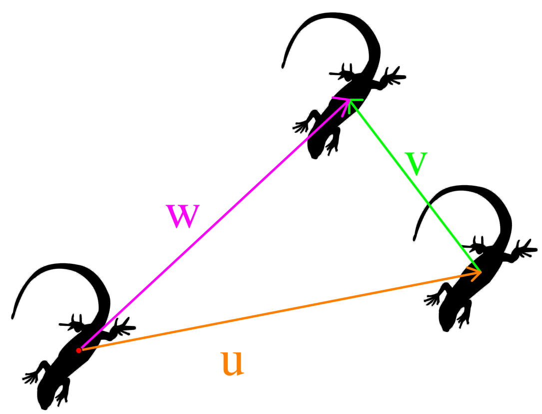
In the previous problem, what happens if the reflections are done in the opposite order, that is, first through DE and then though BC? Explain your work.

What happens if you reflect through THREE parallel mirrors?

Suppose the first two mirrors are distance d apart and the next two are distance e apart. What isometry results from reflecting through the each of the three mirrors in order? Be precise: for example, if the answer is a translation, give the direction and distance of translation, and if the answer is a reflection, give the position of the mirror line, etc. Your answer will be in terms of d and e and the positions of the original mirrors.

Products of isometries

- Suppose we do a translation using vector u, and then do a translation using vector v. We could accomplish the same final result using a single translation through the vector w. So we say that a translation followed by a translation is a translation.



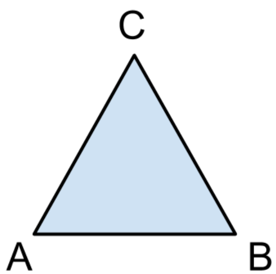
Fill out the following chart. Do the isometry on the left first, then the one on the top.

	Translation	Rotation	Reflection	Glide
Translation				
Rotation				
Reflection				
Glide				

Most squares will have more than one answer. E.g., a reflection of a reflection is a rotation when the mirror lines intersect and a translation if the mirror lines are parallel. Write all possible results and circle the most typical result.

Symmetry Group of the Equilateral Triangle

What are all the symmetries of an equilateral triangle?



That is, what are all the isometries that ...

The triangle has ________ mirror lines and ________ rotations (if we count the do nothing as a rotation by 0°).

List them all.

We will use the following notation for the isometries of the triangle.

- e = do nothing
- r_{120} = rotation by 90° counterclockwise
- r_{240} = rotation by 180° counterclockwise
- $f_{|}$ = flip across a vertical mirror
- $f_{\backslash}$ = flip across diagonal mirror line
- $f_{/}$ = flip across other diagonal mirror line

Fill out the following chart. Do the isometry on the left first, then the one on the top.

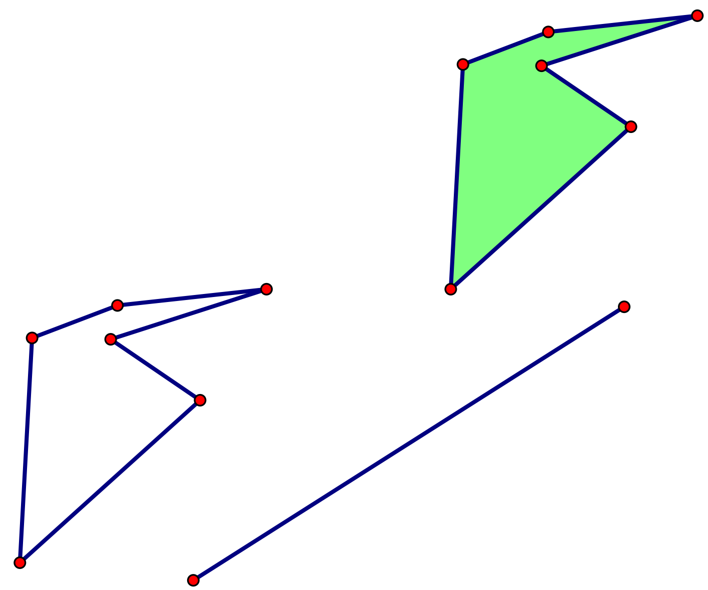
	e	r_{120}	r_{240}	$f_{\vert}$	$f_{\backslash}$	$f_{/}$
e						
r_{120}						
r_{240}						
$f_{\vert}$						
$f_{\backslash}$						
$f_{/}$						

Hint: Think about whether you get a rotation or a reflection. Then follow what happens to a single point, like point A, to determine which rotation or which reflection.

What patterns do you see in this "multiplication table"?

Inverse

The inverse of an isometry undoes the action of the isometry.



Does every isometry have an inverse? If not, find an example of one that does not.

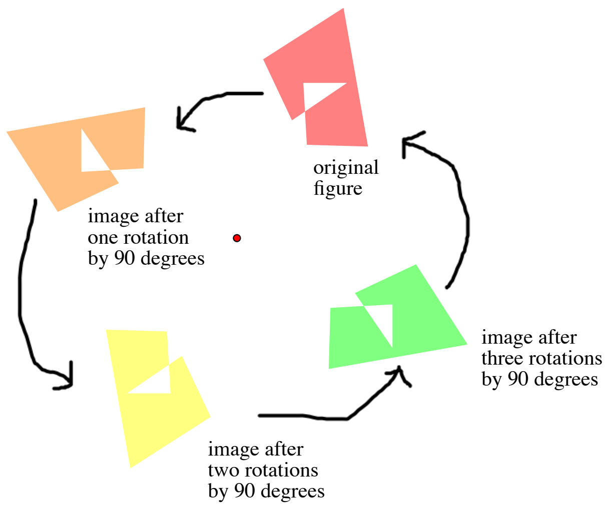
Fill out the following chart:

Isometry	Inverse
Translation in the direction of vector $\vec{v}$	
Rotation by angle θ in the counterclockwise direction	
Reflection across mirror line m	
Glide across mirror m and along vector $\vec{v}$	

Order

The order of an isometry is the number of times you have to repeat it to get back to what you started with. Order is called infinite if you never get back to what you started with.

For example, a 90° rotation has order 4, because when we perform it on a figure four times consecutively, we get back to where we started. Four times consecutively means first perform it on the original figure, then on the resulting image, then on that image, etc.

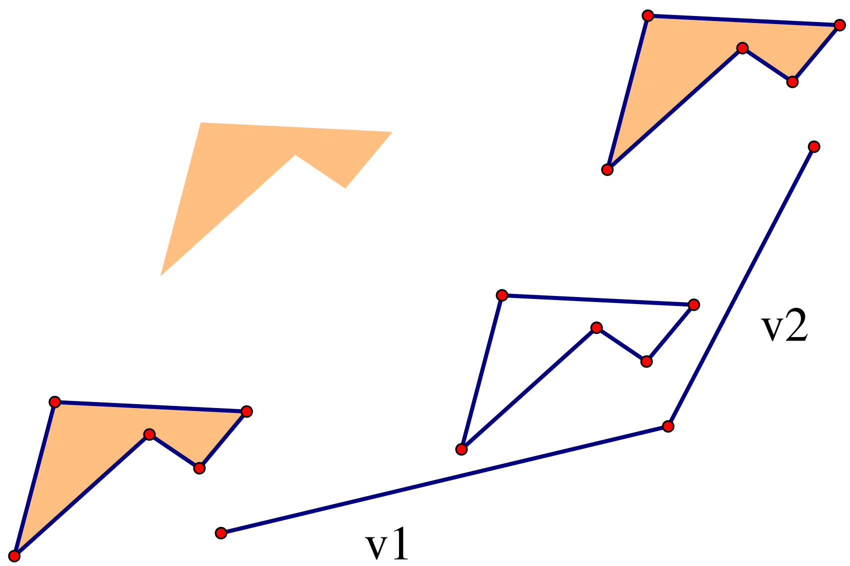


Draw as many conclusions as you can about the order of various isometries. You can use Geogebra to help you.

Isometry	Order
Translation in the direction of vector $\vec{v}$	
Rotation by angle θ in the counterclockwise direction around rotocenter point p	
Reflection across mirror line m	
Glide across mirror m and along vector $\vec{v}$ parallel to m	

Commutativity

Two isometries A and B **commute** if A followed by B gives the same result at B followed by A.

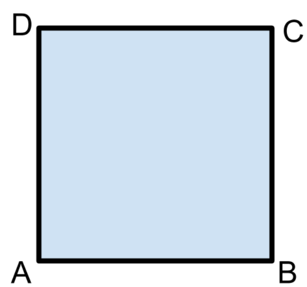


Which isometries do you think commute and which do not?

Note: it may be that, under some circumstances, certain types of isometries commute and under others they do not. Can you describe the circumstances in such cases?

Problems on combining isometries, inverses, order, and commutativity

1. How many rotations, reflections, translations, and glides are isometries of a square?



We will use the following notation for the isometries of the square.

- e = do nothing
- r_{90} = rotation by 90° counterclockwise
- r_{180} = rotation by 180° counterclockwise
- r_{270} = rotation by 270° counterclockwise
- f_{-} = flip across a horizontal mirror
- $f_{|}$ = flip across a vertical mirror
- $f_{\backslash}$ = flip across diagonal mirror line
- $f_{/}$ = flip across other diagonal mirror line

2. Fill out the following chart. Do the isometry on the left first, then the one on the top.

	e	r_{90}	r_{180}	r_{270}	f_-	$f_\vert$	$f_\backslash$	$f_/$
e	e	r_{90}	r_{180}	r_{270}	f_-	$f_\vert$	$f_\backslash$	$f_/$
r_{90}	r_{90}	r_{180}	r_{270}	e	$f_\backslash$	$f_/$	$f_\vert$	f_-
r_{180}								
r_{270}								
f_-								
$f_\vert$								
$f_\backslash$								
$f_/$								

Hint: Think about whether you get a rotation or a reflection. Then follow what happens to a single point, like point A, to determine which rotation or which reflection.

3. What patterns to you see in this "multiplication table"?
4. What are the orders of the following isometries?
   (a) rotation clockwise by 40 degrees
   (b) rotation counterclockwise by 135 degrees
   (c) rotation clockwise by 95 degrees?
   (d) rotation clockwise by any whole number α degrees? any rational number of degrees?
   (e) rotation clockwise by an irrational number of degrees?
   (f) translation due north by 1 cm
   (g) reflection through a horizontal mirror line
5. Give an example of two isometries that do not commute.
6. For the symmetry group of the square, the identity (do nothing) isometry commutes with all the other isometries. Do any of the other 7 isometries commute with all the other isometries in the symmetry group of the square?