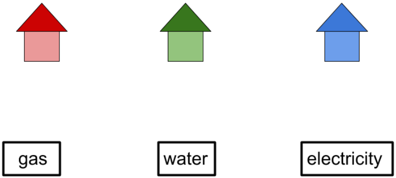
Part XVIII

Euler Characteristic

The goal for this part is to calculate the Euler characteristic of surfaces and use it to prove certain constructions are impossible.

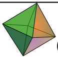
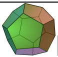
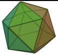
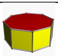
The gas, electricity, and water problem

Suppose there are three cottages on a plane and each needs to be connected to the gas, water, and electricity companies. Using a third dimension or sending any of the connections through another company or cottage is disallowed. Is there a way to make all nine connections without any of the lines crossing each other?



Faces, vertices, and edges of polyhedra

A polyhedron is a 3-dimensional shape with flat polygon faces, straight edges, and sharp corners, called vertices. For example, a cube and a tetrahedron are polyhedra. For each of these polyhedra, count the faces, edges, and vertices.

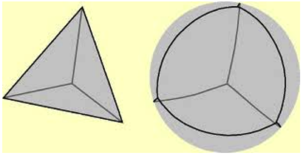
Object	Faces (F)	Edges (E)	Vertices (V)
 Cube			
 Tetrahedron			
 Octahedron			
 Dodecahedron			
 Icosohedron			
 Prism on *n*-sided base			
Pyramid on *n*-sided base 			
Pentagonal Cupola 			
Soccer Ball 			

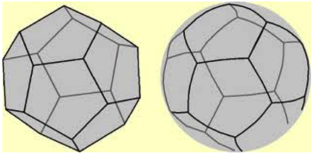
Euler characteristic for polyhedra

- Find a formula relating the number of faces, edges, and vertices of a polyhedra.
- This formula is known as *E*uler's formula and the number ____________ is known as the Euler characteristic.

Euler characteristic for the sphere

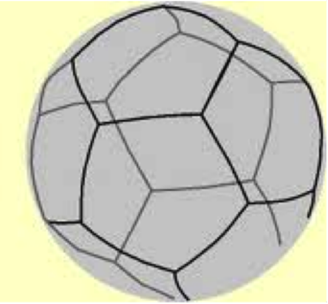
- Note: We don't really need the faces to be flat or the edges to be straight to count them. We could imagine blowing air into a polyhedron and puffing it up like a balloon and we could still count faces, edges, and vertices on the balloon surface.





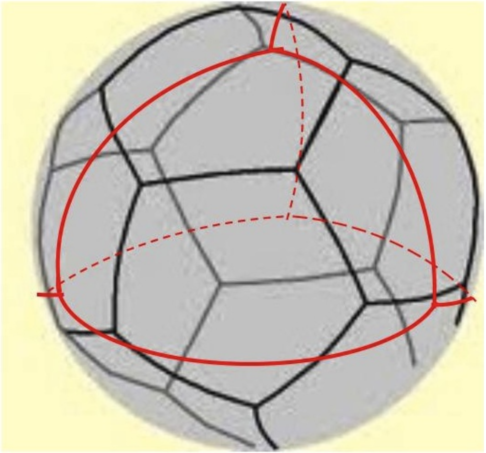
- A network of vertices, edges, and faces on the sphere is called a *map* on the sphere.
- Try to draw a map on the sphere for which Euler's formula does not hold.
- What conditions do we need on the faces and the edges to make sure that Euler's formula always holds for any map on a sphere?

An electric charge proof of Euler's formula for the sphere.



An inking-in proof of Euler's formula

$V - E + F$ using the red lines is the same as $V - E + F$ using the black lines.



Euler characteristic only depends on topology

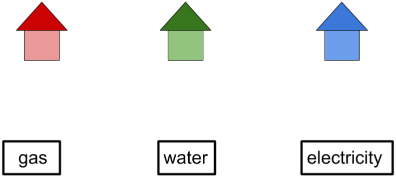
- The inking-in proof works for any topological surface, not just a sphere, to show that $V - E + F$ is the same for any map on that surface.
- Any two polyhedra that have the same topology have the same Euler characteristic.
- The Euler characteristic is called a topological *invariant* of the surface.

Additional proofs of Euler's formula for the sphere can be found at David Eppstein's Geometry Junkyard website.

Look for Twenty Proofs of Euler's Formula $V - E + F = 2$

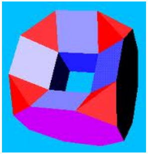
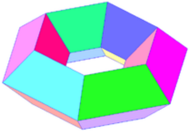
Gas, water, and electricity, revisited

What does Euler characteristic have to do with the gas, water, and electricity problem?



Euler characteristic for other surfaces

Does Euler's formula still hold for the vertices, edges, and faces of a polyhedral torus?

Is it possible to solve the gas, water, and electricity problem on the torus?

Homework

1. Is it possible to solve the houses and utilities problem on a torus? Demonstrate a solution or show that it is not possible.
2. Is it possible to draw 5 points on the plane and connect each pair of points with a line segment in such a way that the line segments do not cross? Prove your answer.
3. Find the Euler characteristic of the following surfaces:
   3.1 the projective plane (Hint: work from the gluing diagram.)
   3.2 the Klein bottle
   3.3 a 3 holed torus
   3.4 a pair of pants
   3.5 external link: the surfaces below