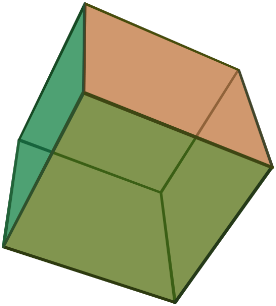
Part XXI

The Gauss-Bonnet Theorem

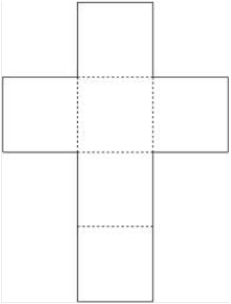
The goal for this part is to state and prove a version of the Gauss-Bonnet Theorem, also known as Descartes Angle Defect Formula. This theorem relates curvature (geometry) to Euler characteristic (topology).

Curvature around the vertices of a cube

Find the angle defect of a piece of the surface of a cube that contains one vertex.

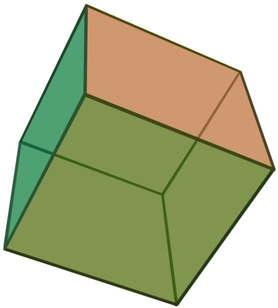


Flattening a cube



Total curvature of a cube

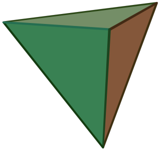
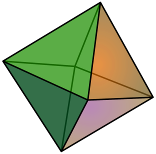
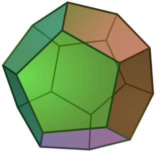
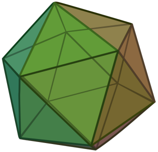
You can find the total curvature (total angle defect) of a surface by dividing it into many small regions and adding up the curvature (angle defect) of each region.

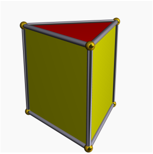
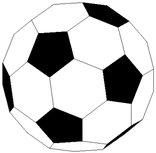
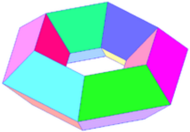
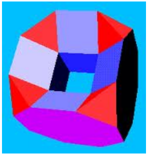


What is the total curvature of a cube?

Total curvature of polyhedra

On a polyhedron, the angle defect of any region that doesn't contain a vertex is 0. Why?
We can calculate the total angle defect of a polyhedron by adding up the angle defects at each vertex.

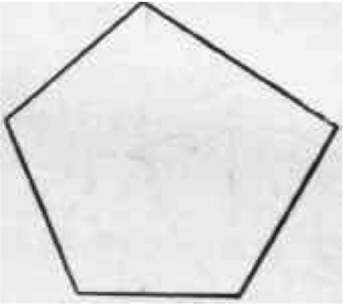
   

What do you notice?

Interior angles of a polygon

What is the sum of the interior angles of a polygon with n sides?



Descartes Angle Defect Formula

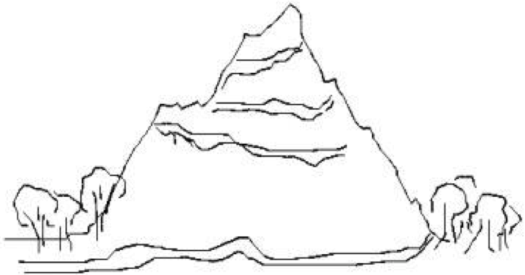
Descartes Angle Defect Formula says:

Why is this so amazing?

- Euler number only depends on the topology of the surface
- But regional curvature (angle defect) depends very much on the geometry of the surface
- Somehow all the ins and outs of regional curvature always exactly balance out so that total curvature ultimately only depends on the topology, not the geometry of the surface

Mountains, earthquakes, and the Gauss-Bonnet Theorem

The Gauss-Bonnet Theorem says that for ANY surface, total curvature $= 2\pi\chi$.



If an earthquake creates a new mountain tomorrow, generating additional positive curvature at the top of the mountain, that new positive curvature has to be exactly balanced by new negative curvature elsewhere.

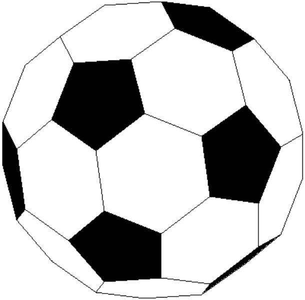
Where is the negative curvature?

Electric charge proof of Descartes Angle Defect Formula

For any map on a given surface, start with a bucket of charge that includes $360\chi = 360V - 360E + 360F$ units of charge.

- $360V$ units of positive charge
- $360E$ units of negative charge
- $360F$ units of positive charge

Distribute the charge on the V vertices, E edges, and F faces of the map.



What is the net charge in each face? How much charge is left in the bucket?

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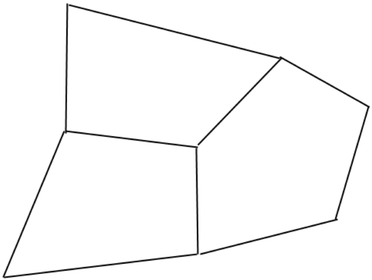
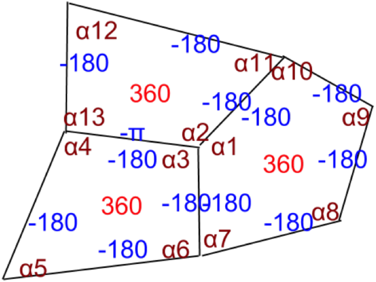


What is the net charge in each face? How much charge is left in the bucket?

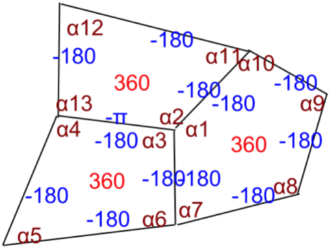
Electric charge proof of Descartes Angle Defect Formula

$360V - 360E + 360F$ units of charge to distribute.

- Put 360 units of charge in the center of each face.
- Put -180 units of charge on each side of each edge.
- For each vertex, place an amount of charge equal to the interior angle α inside each interior angle.
- The charge left over at each vertex is equal to the angle defect.
- So the total charge left in the bucket is the total angle defect.



Electric charge proof of Descartes Angle Defect Formula, Continued



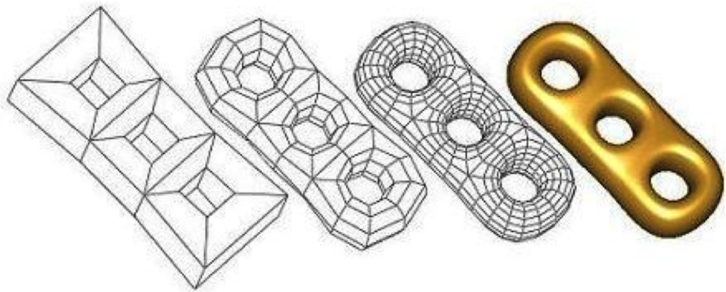
- In each face, the red and blue charges add up to $-180n + 360$, where n is the number of edges in that face.
- But the brown charges in the face add up to $180(n-2)$ since the sum of the interior angles of a polygon is $180(n-2)$.
- So the net charge in each face is 0.
- The net charge of $360V - 360E + 360F$ that we started with is equal to the net charge of 0 on the faces plus the total angel defect left in the bucket.
- $360\chi = \kappa$, the total curvature.

Alternate proof of Descartes Angle Defect Formula

- The angle defect at one vertex is $360 - \sum$ all angles at that vertex.
- The total angle defect $= 360V - \sum$ all angles at all vertices.
- $\sum$ all angles at all vertices $= \sum$ all angles in all faces.
- $\sum$ all angles in one face with n eges is $180(n-2) = 180n - 360$.
- So $\sum$ all angles in all faces $= 180 \sum$ all edges in all faces $-360F = 180 \cdot 2E - 360F = 360E - 360F$.
- So $\sum$ all angles at all vertices $= \sum$ all angles at all faces $= 360E - 360F$.
- So total angle defect $= 360V - (360E - 360F) = 360V - 360F + 360E = 360\chi$.

Gauss-Bonnet Theorem

What about smooth surfaces that are not polyhedra?

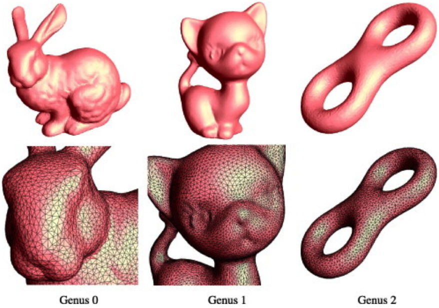


Reference

image: CGAL, Computational Geometry Algorithms Library, http://www.cgal.org

- We can approximate any smooth surface with polyhedra.
- A limiting argument shows that total curvature $= 2\pi\chi$ for any topological surface.
- This is called the Gauss-Bonnet Theorem.
- The Gauss-Bonnet Theorem and Descartes Angle Defect say essentially the same thing: one is for smooth surfaces and one is for polyhedra.

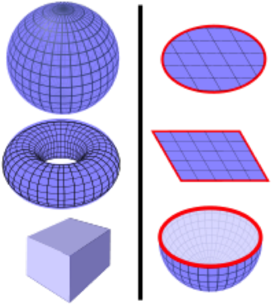
Polyhedral approximations to surfaces



From Wei Zeng, Ren Guo, Feng Luo, Xianfeng Gu, Discrete heat kernel determines discrete Riemannian metric, Graphical Models, Volume 74, Issue 4, July 2012

Gauss-Bonnet for surfaces with boundary

So far, we have only considered total curvature for surfaces without boundary



- Is there a Gauss-Bonnet Theorem for surfaces with boundary?
- State and prove Descartes Angle Defect Formula for surfaces with boundary
- Be careful about how you define angle defect for vertices on the boundary

Proof of Gauss-Bonnet for surfaces with boundary

- Start with 360χ units of charge in a bucket. This is $360F - 360E + 360F$ units of charge.
- Describe how you would place the charge on the surface with boundary, and illustrate with the surface (with boundary) formed by a truncated square pyramid without a base.
  - Where do the $360F$ units of charge go?
  - Where do the $360E$ units of negative charge go?
  - How do you distribute the $360V$ units of charge? Hint: for each vertex on the boundary, start by putting 180 units of charge on the outside of the surface near the vertex.
- How much charge is left in the bucket, in terms of the angle defect?
- For each face, use an equation to explain why the total charge in that face is 0.
- There is still some charge lying around the "coastline", just outside the boundary of the surface. Explain why that charge adds up to 0.
- Explain why $360\chi = \kappa$, the total angle defect.

Homework

1. Give a precise statement of the Descartes Angle Defect Theorem (a.k.a. the Gauss-Bonnet Theorem) for surfaces with boundary. Specify how angle defect is calculated along the boundary, especially at corners of the boundary.
2. Verify Descartes Angle Defect Theorem for a surface that is a flat equilateral triangle. By verify, I mean, calculate the Euler number and the total curvature , counting carefully along the boundary vertices, and check that these quantities have the correct predicted relationship.
3. Find the total curvature of the following surfaces:
   3.1 A Klein bottle.
   3.2 A Mobius band.
   3.3 A cylinder, without the top and bottom.
   3.4 A disk (that is, a surface that is the inside of a circle).

More Homework

4 Prove Descartes Angle Defect Theorem for surfaces with boundary. Hint: draw a picture and distribute 360V - 360E + 360F units of charge to the vertices, edges, and faces like we did in class. This time, some of the edge charge will lie outside of the figure. Figure out how it gets canceled to leave 0 net charge on the figure and the total angle defect of charge on the surface. It is a little different around the boundary, so you can't completely copy what we did in class.