Part XXIII

Orbifolds and the Magic Theorem

The goal for this part is to justify the Magic Theorem by analyzing orbifolds.

Topological structure of orbifolds

We have seen that orbifolds made from wallpaper patterns have to have positive or zero Euler number. Therefore, any wallpaper orbifold must be one of the following surfaces:

- sphere
- sphere with one puncture (disk)
- sphere with two punctures (cylinder)
- torus
- projective plane
- projective plane with one puncture (Mobius band)
- Klein bottle

We will consider each of these cases.

The Sphere

If the orbifold is a topological sphere ...

- Can there be mirror lines or kaleidoscope points in the wallpaper pattern?
- Can there be gyration points in the wallpaper pattern?
- How does the curvature of a cone point compare to the cost of the corresponding gyration point?
- What does the Gauss-Bonnet Theorem say in this context?
- Restate the Gauss-Bonnet Theorem in the language of the Magic Theorem.

The Disk

If the orbifold is a topological disk ...

- Can there be mirror lines or kaleidoscope points in the wallpaper pattern?
- Can there be gyration points in the wallpaper pattern?
- How does the curvature of a cone point compare to the cost of the corresponding gyration point?
- How does the curvature of boundary corner compare to the cost of the corresponding kaleidoscope point?
- What does the Gauss-Bonnet Theorem say in this context?
- Restate the Gauss-Bonnet Theorem in the language of the Magic Theorem.

The Cylinder

If the orbifold is a topological cylinder ...

- Can there be kaleidoscope points in the wallpaper pattern?
- Can there be gyration points in the wallpaper pattern?
- How many mirror lines must there be?
- What must the signature be?
- Verify that the Magic Theorem hold in this situation.

The Torus

If the orbifold is topologically a torus...

- Can there be kaleidoscope points in the wallpaper pattern?
- Can there be gyration points in the wallpaper pattern?
- Can there be any X's or $\star$'s in the wallpaper signature?
- What signature does this correspond to?
- Why does the Magic Theorem hold in this situation?

The Projective Plane

If the orbifold is topologically a projective plane ...

- Can there be mirror lines or kaleidoscope points in the wallpaper pattern?
- Can there be gyration points in the wallpaper pattern?
- Is there an X in the wallpaper signature?
- How does the curvature of a cone point compare to the cost of the corresponding gyration point?
- What does the Gauss-Bonnet Theorem say in this context?
- Restate the Gauss-Bonnet Theorem in the language of the Magic Theorem.

The Mobius Band

If the orbifold is topologically a Mobius band ...

- Can there be kaleidoscope points in the wallpaper pattern?
- Can there be gyration points in the wallpaper pattern?
- Is there an X in the wallpaper signature?
- How many mirror lines must there be?
- What must the signature be?
- Verify that the Magic Theorem holds in this situation.

The Klein Bottle

If the orbifold is topologically a Klein bottle...

- Can there be mirror lines or kaleidoscope points in the wallpaper pattern?
- Can there be gyration points in the wallpaper pattern?
- Is there an X in the wallpaper signature?
- What must the signature be?
- Verify that the Magic Theorem holds in this situation.

Homework

1. Explain why the Magic Theorem holds for patterns whose orbifold is a Mobius band. You can use the questions on today's class notes as a guide – find the page on the Mobius band towards the end.
2. Please bring balls with patterns on them (or any other interesting 3d objects) to class.