Part IV

Four (or Five) Types of Isometries

Goal: Prove that there are only four (or five) isometries of the plane:

1.
2.
3.
4.
Polygons that share three vertices

A *vertex* of a polygon is a corner where two edges meet.

These two polygons share two vertices.

- Is it possible to draw two congruent polygons in the plane that share three corresponding vertices but don’t share all their vertices?
- What about for polyhedra in 3-dimensional space?
Theorem

*If two congruent polygons share three corresponding vertices that are not collinear, then they share all vertices.*
Two congruent polygons that share three corresponding vertices (that are not colinear) share all vertices.

Proof.

- Suppose we have two congruent polygons that share three corresponding vertices \( A, B, \) and \( C. \)
- If there is a fourth vertex \( D \) on the first vertex, figure out how far it is from each of \( A, B, \) and \( C. \) Say it is \( a \) units from \( A, b \) units from \( B, \) and \( c \) units from \( C. \)
- Vertex \( D \) must lie on the intersection of the three circles centered at \( A, B, \) and \( C \) of radii \( a, b, \) and \( c, \) respectively.
- As long as \( A, B, \) and \( C \) are not colinear, then there is only one point of intersection of the three circles.
- Since vertex \( D' \) on the second polygon is also distance \( a \) from \( A, \) distance \( b \) from \( B, \) and distance \( c \) from \( C, \) it must also be on the intersection of the three circles.
- So vertex \( D' \) on the second polygon must coincide with vertex \( D \) on the first polygon.
There are no other isometries out there

There are no other isometries of the plane besides:
- translations
- reflections
- rotations
- glide reflections
- the identity

How do we know?

What about in 3-d?
Translations, rotations, reflections, and glides are the only isometries of the plane.

Proof.

- Three vertices of a triangle $ABC$ can be brought onto three vertices of an isometric triangle $A'B'C'$ that using a product of one, two, or three reflections. How?
Translations, rotations, reflections, and glides are the only isometries of the plane.

Proof.

- Three vertices of a triangle $ABC$ can be brought onto three vertices of an isometric triangle $A'B'C'$ that using a product of one, two, or three reflections. How?
- If an isometry agrees with a product of reflections on three (non-collinear) points $A$, $B$, and $C$, then it agrees with the product of reflections on all points. Why?
Translations, rotations, reflections, and glides are the only isometries of the plane.

Proof.

- Three vertices of a triangle $ABC$ can be brought onto three vertices of an isometric triangle $A'B'C'$ that using a product of one, two, or three reflections. How?
- If an isometry agrees with a product of reflections on three (non-collinear) points $A, B, \text{and } C$, then it agrees with the product of reflections on all points. Why?
- Therefore, any isometry is a product of one, two, or three reflections.
Translations, rotations, reflections, and glides are the only isometries of the plane.

Proof.

- Three vertices of a triangle \(ABC\) can be brought onto three vertices of an isometric triangle \(A'B'C'\) that using a product of one, two, or three reflections. How?

- If an isometry agrees with a product of reflections on three (non-collinear) points \(A, B,\) and \(C\), then it agrees with the product of reflections on all points. Why?

- Therefore, any isometry is a product of one, two, or three reflections.

- If only one reflection is needed, then the isometry is a reflection.
Translations, rotations, reflections, and glides are the only isometries of the plane.

Proof.

- Three vertices of a triangle $ABC$ can be brought onto three vertices of an isometric triangle $A'B'C'$ that using a product of one, two, or three reflections. How?
- If an isometry agrees with a product of reflections on three (non-collinear) points $A$, $B$, and $C$, then it agrees with the product of reflections on all points. Why?
- Therefore, any isometry is a product of one, two, or three reflections.
- If only one reflection is needed, then the isometry is a reflection.
- If exactly two reflections are needed, then the isometry is:
- If exactly three reflections are needed, then the isometry is:
- Why is the last statement true?
Why is a product of three reflections always a reflection or a glide reflection?
Proof that product of three reflections always a reflection or a glide reflection, part 1

- Suppose we have an isometry that is a product of three reflections through mirrors \( m_1 \), \( m_2 \), and \( m_3 \).
Proof that product of three reflections always a reflection or a glide reflection, part 1

- Suppose we have an isometry that is a product of three reflections through mirrors \( m_1, m_2, \) and \( m_3 \).
- If \( m_1, m_2, \) and \( m_3 \) are all parallel, then the product of the reflections is a __________________________. Why?
Proof that product of three reflections always a reflection or a glide reflection, part 1

- Suppose we have an isometry that is a product of three reflections through mirrors \( m_1, m_2, \) and \( m_3 \).
- If \( m_1, m_2, \) and \( m_3 \) are all parallel, then the product of the reflections is a _______________________. Why?
- If \( m_2 \) and \( m_3 \) intersect, then reflection through \( m_2 \) and then \( m_3 \) is a rotation with rotocenter at the intersection of \( m_2 \) and \( m_3 \).
- So we can think of our isometry as reflection through \( m_1 \) followed by rotation around this intersection point.
Proof that product of three reflections always a reflection or a glide reflection, part 1

- Suppose we have an isometry that is a product of three reflections through mirrors $m_1$, $m_2$, and $m_3$.
- If $m_1$, $m_2$, and $m_3$ are all parallel, then the product of the reflections is a ______________ . Why?
- If $m_2$ and $m_3$ intersect, then reflection through $m_2$ and then $m_3$ is a rotation with rotocenter at the intersection of $m_2$ and $m_3$.
- So we can think of our isometry as reflection through $m_1$ followed by rotation around this intersection point.
- But if we rotate $m_2$ and $m_3$ around this intersection point, we’ll still get the same rotation with the same rotocenter.
- So rotate $m_2$ and $m_3$ around their intersection point until $m_2$ is perpendicular to $m_1$. 
Proof that product of three reflections always a reflection or a glide reflection, part 2

▶ Now our isometry is reflection through \( m_1 \), then \( m_2 \), then \( m_3 \), and \( m_1 \) is perp to \( m_2 \).
Proof that product of three reflections always a reflection or a glide reflection, part 2

- Now our isometry is reflection through \( m_1 \), then \( m_2 \), then \( m_3 \), and \( m_1 \) is perp to \( m_2 \).
- So reflection through \( m_1 \) then \( m_2 \) is the same as rotation by [fill in degrees] degrees with rotocenter at the intersection of \( m_1 \) and \( m_2 \).
Proof that product of three reflections always a reflection or a glide reflection, part 2

- Now our isometry is reflection through \( m_1 \), then \( m_2 \), then \( m_3 \), and \( m_1 \) is perp to \( m_2 \).
- So reflection through \( m_1 \) then \( m_2 \) is the same as rotation by \( \frac{180}{2} \) degees with rotocenter at the intersection of \( m_1 \) and \( m_2 \).
- If we rotate \( m_1 \) and \( m_2 \) around their intersection point, we still get the same rotation.
- So rotate \( m_1 \) and \( m_2 \) around their intersection point until \( m_2 \) is parallel to \( m_3 \).
Proof that product of three reflections always a reflection or a glide reflection, part 2

- Now our isometry is reflection through $m_1$, then $m_2$, then $m_3$, and $m_1$ is perp to $m_2$.
- So reflection through $m_1$ then $m_2$ is the same as rotation by $\text{___________}$ degrees with rotocenter at the intersection of $m_1$ and $m_2$.
- If we rotate $m_1$ and $m_2$ around their intersection point, we still get the same rotation.
- So rotate $m_1$ and $m_2$ around their intersection point until $m_2$ is parallel to $m_3$.
- Now we have $m_2$ and $m_3$ parallel, and $m_1$ perpendicular to both.
Proof that product of three reflections always a reflection or a glide reflection, part 2

- Now our isometry is reflection through $m_1$, then $m_2$, then $m_3$, and $m_1$ is perp to $m_2$.
- So reflection through $m_1$ then $m_2$ is the same as rotation by ___________ degrees with rotocenter at the intersection of $m_1$ and $m_2$.
- If we rotate $m_1$ and $m_2$ around their intersection point, we still get the same rotation.
- So rotate $m_1$ and $m_2$ around their intersection point until $m_2$ is parallel to $m_3$.
- Now we have $m_2$ and $m_3$ parallel, and $m_1$ perpendicular to both.
- This means we reflect through $m_1$ and then translate in direction of $m_1$.
- This is exactly a glide reflection!!
Proof that product of three reflections always a reflection or a glide reflection, technical details

There are a few small details to worry about

- If \( m_1 \) and \( m_2 \) and \( m_3 \) all intersect in the same point, then we need to modify the argument:
  - Rotate \( m_2 \) and \( m_3 \) around their intersection point until \( m_2 \) is on top of \( m_1 \).
  - Then we have reflection through the same mirror twice, followed by rotation through \( m_3 \).
  - Reflecting through the same mirror twice does nothing.
  - So this is just a reflection through \( m_3 \).

- If \( m_2 \) and \( m_3 \) are parallel, we need to modify the argument:
  - Start by rotating \( m_1 \) and \( m_2 \) around their intersection point until \( m_2 \) intersects \( m_3 \) and continue as before.
Homework

1. Give an algorithm (step by step instructions) for how to write ANY isometry as a product of at most three reflections.