Part IX

Finite figures

The goal for this part is relate “rosettes” and “finite figures”.

Finite figures

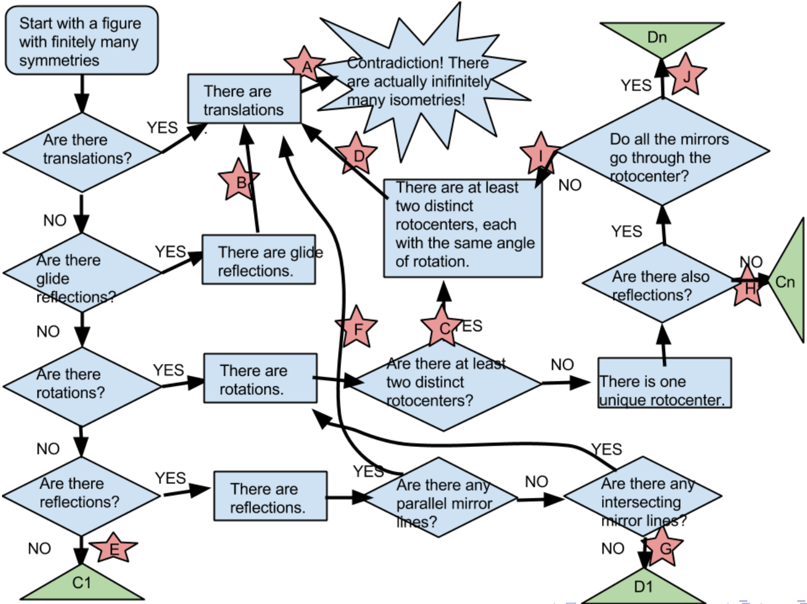
Rosettes are examples of *finite figures*.
What is finite about them?

Other finite figures

Definition: a *finite figure* is a figure that has only finitely many symmetries; that is, finitely many isometries that preserve the figure. Are there any finite figures that are NOT rosettes (that is, not of the type C_n or D_n)?

Finite figures are always rosettes: flowchart

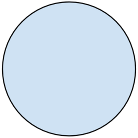
Fill in explanations for each star in the flow chart.



Finite figures are always rosettes: dialog, part 1

The following dialog, once completed, will give a thorough explanation, or proof, of why there are no other finite figures besides rosettes. Please rename Character 1 and Character 2 and fill in Character 2's explanations. Please also add figures to clarify. You are encouraged to modify Character 1's lines also.

- Character 1: I heard you say that any figure with finitely many symmetries has to have symmetry type C_n or D_n. But I don't believe it. I bet I can come up with an example of a figure with finitely many symmetries that is NOT of type C_n or D_n.
- Character 2:
- Character 1: Well, first of all, what about circle? That isn't really a C_n or a D_n figure. It can't be. It doesn't have any petals.

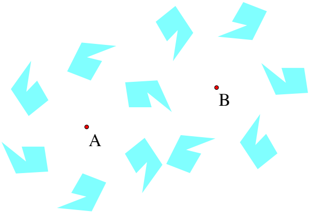


- Character 2:
- Character 1: Okay, maybe a circle isn't a good example then. How about this picture. It's not a C_n or D_n pattern, but it only has **one** symmetry: translation. And last I checked, **one** is a finite number.

- Character 2:
- Character 1: Okay, fine, so figures with translation are out. Let me see if I can come up with an example of a figure with only finitely many symmetries using glide reflection ...
- Character 2:

Finite figures are always rosettes: dialog, part 2

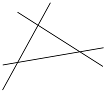
- Character 1: No problem. I don't need translation or glide reflection to show that you're wrong. I'll just work with rotations and reflections and find an example of a figure with finitely many symmetries that isn't type C_n or D_n.
- Character 2:
- Character 1: Maybe I should look for a figure that has a rotation through a crazy, irrational angle like $\sqrt{2}$ degrees.
- Character 2:
- Character 1: So, you want me to stick with rational angles. Ok, whatever. Let me draw a figure here with two different rotocenters. THAT won't be C_n or D_n since C_n and D_n only have one rotocenter. Here, I've got one: this figure has a rotation of order 6 around point A and another rotation of order 6 around B. Look – finitely many isometries – all rotations.



- Character 2: (Hint: what happens when you follow a 60° clockwise rotation around one point by a 60° rotation counterclockwise around a different point? What isometry results? Once you figure this out, write some lines in for Character 2.)
- Character 1: Oh yeah, that's true. Ok, what happens if we do an order 6 rotation around A and an order 5 rotation around B?
- Character 2: Hint: if we rotate the rotocenter A around the rotocenter B, we get another rotocenter A'. This gives two rotocenters A and A', both of which have the same order. Fill in some words for Character 2 around this.

Finite figures are always rosettes: dialog, part 3

- Character 1: Well you got me there. Having two different rotocenters seems to always produce a translation, and once there's a translation, there are always infinitely many symmetries. So, let's work with rotation and reflection instead. Let me draw a figure that has rotational symmetry around a point A, and reflection symmetry through a mirror line M, but here's the catch – I'm going to make M NOT go though A. That way, it won't be a C_n or D_n type pattern, even though it will have finitely many reflections and rotations.
- Character 2:
- Character 1: Wow, good thing I don't get frustrated easily. Okay, so it won't work to have rotation plus reflection on a mirror line that doesn't contain the rotocenter. Let me just try reflections alone. I could try two reflections through parallel mirror lines. That won't be C_n or D_n.
- Character 2:
- Character 1: So let me try two reflections through mirror lines that intersect then!
- Character 2:
- Character 1: Okay, maybe three or more reflections through different mirror lines. I won't make any of the mirror lines parallel, since I don't want to create any translations. But I won't make them all three intersect at the same point, since I am trying to avoid D_n symmetry. Something like this configuration of mirror lines. Tada! A figure with finitely many symmetries that can't be C_n or D_n.



- Character 2:

Finite figures are always rosettes: dialog, part 4

- Character 1: Okay, let me review a little. You've convinced me that anything with translation symmetry will have infinitely many symmetries (even though they're all closely related and intensely boring). You've also convinced me that anytime there are two different rotocenters, there's a translation. And anytime there is a mirror line and a rotocenter that doesn't lie on top of the mirror, there have to be two different rotocenters, and therefore a translation. This eliminates all arrangements of mirrors and rotocenters except for a single rotocenter that lies on top of all mirrors, or no rotocenter at all. If there is a unique rotocenter with all mirrors (if there are any) going through it, then we have D_n, or else C_n if there are no mirrors. If there is no rotocenter at all, we can only have at most one mirror, so we have D_1, or C_1 if there are no mirrors.
- Character 2:
- Character 1: So I guess there really is no way to create a figure with finitely many symmetries besides C_n and D_n figures. Finitely many symmetries is the same thing as being a C_n or D_n symmetry. The same thing as a rosette. Who would have thought.
- Character 2:
- Character 1: But wait! I just thought of something! What if we work in 3-dimensional space instead of 2-dimensional space! I bet I can prove you wrong then!
- Character 2:

Homework

1. Prove that finite figures are all of type Dn or Cn. You can do this by EITHER filling in the dialog, OR by explaining the reasoning at each starred step in the flow chart. Or, if you'd like, you can submit a more traditional mathematical proof.
2. Also (ungraded) submit a frieze pattern that you like via the Sakai Assignments tab. Frieze patterns have translation symmetry along one direction.