Polar Coordinates

After completing this section, students should be able to:

• Graph a point given in polar coordinates.
• Explain how to graph a point given in polar coordinates when the given r-value is negative.
• Find more than one way to represent a point in polar coordinates.
• Convert points from polar coordinates to rectangular coordinates and vice versa.
• Convert equations from rectangular coordinates to polar coordinates.
• Graph simple equations in polar coordinates, such as $\theta = 2$, $r = 3$, $r \sin \theta = 5$, $r = \cos \theta$.
• Match graphs and equations in polar coordinates.
• Predict the symmetry of a graph from its equation in polar coordinates.
Cartesian coordinates: \((x, y)\)
Polar coordinates: \((r, \theta)\), where \(r\) is:

and \(\theta\) is:

**Example.** Plot the points, given in polar coordinates.

1. \((8, -\frac{2\pi}{3})\)

2. \((5, 3\pi)\)

3. \((-12, \frac{\pi}{4})\)

**Note.** A negative angle means to go clockwise from the positive x-axis. A negative radius means jump to the other side of the origin, that is, \((-r, \theta)\) means the same point as \((r, \theta + \pi)\)
Note. To convert between polar and Cartesian coordinates, note that:

- \( x = ______ \)
- \( y = ______ \)
- \( r = ______ \)
- \( \tan \theta = ______ \)

Example. Convert \((5, -\frac{\pi}{6})\) from polar to Cartesian coordinates.

Example. Convert \((-1, -1)\) from Cartesian to polar coordinates.
**Review.** Points on the plane can be written in terms of rectangular coordinates (a.k.a. Cartesian coordinates) \((x, y)\) or in terms of polar coordinates \((r, \theta)\) where \(r\) represents ...

and \(\theta\) represents ...

The quantities \(x\) and \(y\) and \(r\) and \(\theta\) are related by the equations ...

**Review.** Find the rectangular coordinates of a point with the polar coordinates \((5, \frac{7\pi}{4})\)

A. \((\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\)

B. \((-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})\)

C. \((\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2})\)

D. \((-\frac{5\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2})\)
Review. Find the polar coordinates of a point with rectangular coordinates \((-2, -2 \sqrt{3})\). More than one answer may be correct.

A. \(\left(1, \frac{\pi}{3}\right)\)

B. \(\left(4, \frac{\pi}{3}\right)\)

C. \(\left(4, \frac{4\pi}{3}\right)\)

D. \(\left(4, -\frac{2\pi}{3}\right)\)

E. \(\left(-4, \frac{\pi}{3}\right)\)
Example. Plot the following curves and rewrite using Cartesian coordinates.

1. \( r = 7 \)

2. \( \theta = \frac{\pi}{3} \)
Example. Plot the following curves and rewrite the first one using Cartesian coordinates.

3. \( r = 12 \cos(\theta) \)

4. \( r = 6 + 6 \cos(\theta) \) (an example of a limacon)
Example. Convert the Cartesian equations to polar coordinates:

1. \( y = -2 \)

2. \( y = x \)

3. \( y^2 + (x - 5)^2 = 25 \)

4. \( 4y^2 = x \)
Example. Match the polar equations with the graphs. See how many you can guess without using graphing software, then use graphing software to help.

1. \( r = 5 \cos \theta \)
2. \( r = 5 - 5 \sin \theta \)
3. \( r = 5 - 8 \sin \theta \)
4. \( r = 5 \cos 2\theta \)
5. \( r = 5 \cos 3\theta \)
6. \( r^2 = 5 \cos 2\theta \)
Parametric Equations

After completing this section, students should be able to:

- Draw the graph for a curve described with parametric equations by plotting $x$ and $y$ values corresponding to various $t$ values.
- Convert from parametric equations to a rectangular (Cartesian) equation.
- Convert from a Cartesian equation to parametric equations.
- Write the equation for a circle or ellipse in parametric equations.
- Write the equation for a line in parametric equations.
Definition. A cartesian equation for a curve is an equation in terms of $x$ and $y$ only.

Definition. Parametric equations for a curve give both $x$ and $y$ as functions of a third variable (usually $t$). The third variable is called the parameter.

Example. Graph $x = 1 - 2t$, $y = t^2 + 4$

\[
\begin{array}{c|c|c}
  t & x & y \\ 
  \hline 
  -2 & 5 & 8 \\
  -1 & 3 & 5 \\
  0 & & \\
\end{array}
\]

Find a Cartesian equation for this curve.
Example. Plot each curve and find a Cartesian equation:

1. \( x = \cos(t), \ y = \sin(t), \) for \( 0 \leq t \leq 2\pi \)
2. \( x = \cos(-2t), \ y = \sin(-2t), \) for \( 0 \leq t \leq 2\pi \)
3. \( x = \cos^2(t), \ y = \cos(t) \)
Example. Write the following in parametric equations:

1. \( y = \sqrt{x^2 - x} \) for \( x \leq 0 \) and \( x \geq 1 \)

2. \( 25x^2 + 36y^2 = 900 \)
Example. Describe a circle with radius $r$ and center $(h, k)$:

a) with a Cartesian equation

b) with parametric equations
**Review.** Cartesian equations are ...

Parametric equations ...

**Review.** Which of the following graphs represents the graph of the parametric equations $x = \cos t$, $y = \sin t$. (The horizontal axis is the $x$-axis and the vertical axis is the $y$-axis.)

A.  

B.  

C.
Example. Find a Cartesian equation for the curve and graph the curve with an arrow to represent direction of motion.

1. $x = 3 + 2t, \quad y = -5 - 4t, \quad -2 \leq t \leq 2$

Methods:

2. $x = 1 + 3 \cos(t) + 3, \quad y = 1 + 4 \sin(t)$

3. $x = e^{2t}, \quad y = e^{-4t}$
Example. Find parametric equations for the curve.

1. \( x = -y^2 - 6y - 9 \)

2. \( \frac{x^2}{9} + \frac{y^2}{49} = 1 \)

3. \( 4(x - 2)^2 + 25(y + 1)^2 = 100 \)
Extra Example. What is the equation for a circle of radius 8 centered at the point \((5, -2)\)

1. in Cartesian coordinates?

2. in parametric equations?
Extra Example. Find parametric equations for a line through the points \((2, 5)\) and \((6, 8)\).

1. any way you want.

2. so that the line is at \((2, 5)\) when \(t = 0\) and at \((6, 8)\) when \(t = 1\).
Difference Quotient

After completing this section, students should be able to:

- Explain what a difference quotient represents in terms of a graph of a function or in a context like when the function represents distance travelled over time.
- Calculate and simplify the difference quotient of a various functions, including linear, quadratic, rational, and square root functions.
For a function \( y = f(x) \),

**Definition.** A secant line is

**Definition.** The average rate of change for \( f(x) \) on the interval \([a, b]\) is

**Example.** The average rate of change for \( f(x) = \sqrt{x} \) on the interval \([1, 4]\) is
Definition. A difference quotient represents
Example. Find and simplify the difference quotient for $f(x) = 2x^2 - x + 3$
Review. Which of the following statements are true?

A. The average rate of change of a function $f(x)$ on the interval $[a, b]$ is given by the formula $\frac{f(b) - f(a)}{b - a}$.

B. The average rate of change of a function represents the slope of a secant line.

C. The difference quotient represents an average rate of change.

D. The difference quotient is given by the formula $\frac{f(x) - f(h)}{x - h}$.
Example. Find the average rate of change of \( f(x) = 2x^2 - 2x + 3 \) from \( x = -2 \) to \( x = 1 \).
Example. Find the difference quotient for $f(x) = 2x^2 - 2x + 3$
Example. Find and simplify the difference quotient for $f(x) = \frac{5}{x - 6}$
Extra Example. Find and simplify the difference quotient for $f(x) = \sqrt{x} + 4$. 