Math 231L Calculus co-req

Fall 2020

1 Rationalizing Numerators and Denominators and Simplifying Complex Fractions

In calculus, you will be asked to compute limits like $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$ that you can't compute just by plugging in 4 for x.

To do these problems, you will need to rewrite the expression by rationalizing the numerator, which means rewriting so that there are no square roots in the numerator.

To rationalize the numerator, you multiply the both numerator and the denominator by the conjugate of the numerator.

Example: Find the conjugate of:

1. $\sqrt{a} + \sqrt{b}$
2. $5 + \sqrt{y}$
3. $\sqrt{x} - 2$

Warm-up Problem 1: Rationalize the numerator for $\frac{\sqrt{x}-2}{x-4}$

Tip: When simplifying by rationalizing the numerator, it is best to leave the denominator in factored form rather than multiplying out. That way you'll be able to see and cancel common factors more easily. You will still want to multiply out the numerator.

In calculus, you will be asked to compute limits like $\lim_{x\to 5} \frac{\frac{1}{5} - \frac{1}{x}}{5 - x}$.

Since you can't plug in $x = 5$ here (why not?), the trick will be to rewrite the complex fraction and try to cancel out the parts that are causing trouble by going to 0 when you try to plug in $x = 5$.

Warm-up Problem 2: Simplify the fraction $\frac{\frac{1}{5} - \frac{1}{x}}{5 - x}$

Problems:

1. Rationalize the numerator: $\dfrac{\sqrt{9+h}-3}{h}$

2. Simplify the complex fraction. $\dfrac{\frac{1}{2}+\frac{2}{x}}{4+x}$

3. Simplify the complex fraction. $\dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$

Extra Problems:

4. Rationalize the numerator: $\dfrac{\sqrt{x^2+9}-5}{x+4}$

5. Simplify the complex fraction: $\dfrac{(3+h)^{-1}-3^{-1}}{h}$

6. Simplify the complex fraction *and then* simplify further by rationalizing the numerator!

$$\frac{2-\dfrac{2}{\sqrt{a}}}{1-a}$$

ALEKS topics related to rationalizing the numerator

1. Simplifying a product involving square roots using the distributive property: Advanced
2. Rationalizing a denominator using conjugates: Variable in denominator

ALEKS topics related to simplifying complex fractions

1. Complex fraction without variables: Problem type 1, 2
2. Complex fraction involving multivariate monomials
3. Complex fraction: Quadratic factoring
4. Complex fraction made of sums involving rational expressions: Problem type 1, 2, 3, 4, 6
5. Complex fraction made of sums involving rational expressions: Multivariate

Videos on rationalizing the numerator

- None yet, although there are a couple examples of rationalizing the denominator at the end of the snow day video in the Precalculus playlist: Calculus 1 Playlist > Simplifying Radicals - Snow Day Examples

Videos on complex fractions

- Calculus 1 Coreq Playlist > Rational Expressions
- Calculus 1 Coreq Playlist > Difference Quotient

2 Lines and Rational functions

In Calculus, you will be asked to find the equation of a tangent line. To do this, you need to know how to find the equation of a line given its slope and a point on the line.

- *Slope intercept form* means the form $y = mx + b$.
  - The *slope* is m.
  - The *y-intecept* is b.
- Point slope form means the form $(y - y_0) = m(x = x_0)$.
  - (x_0, y_0) is a point on the line.
  - m is the slope

Warm-up problem 1: Find the equation of a line with slope $-\frac{2}{3}$ that goes through the point $(1, 5)$.

In calculus, you will evaluate limits of rational functions. The horizontal asymptotes of a function correspond to the "end behavior" or the limit of the function as $x \to \infty$ or $x \to -\infty$. These are called limits AT infinity because x is going to infinity.

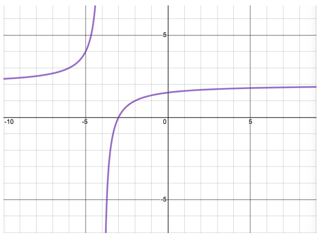
The vertical asymptotes are where the function's y-values shoot off to ∞ or $-\infty$ as x goes to a value. These are called limits OF infinity because y is going to infinity.

Let's review rational functions with an eye towards horizontal asymptotes (limits AT infinity) and vertical asymptotes (limits OF infinity).

- The *holes* correspond to the x-values for factors that make both the numerator and denominator 0, but cancel out.
  - For $f(x) = \dfrac{2(x-1)(x+3)}{(x+4)(x-1)}$, there is a hole at $x = 1$
- The *vertical asymptotes* correspond to x-values that make the denomiantor 0 (but those factors don't cancel out on the numerator).
  - For $f(x) = \dfrac{2(x-1)(x+3)}{(x+4)(x-1)}$, there is a vertical asymptote at $x = -4$
- You can find the *horizontal asymptotes* by comparing the degrees and leading terms on the numerator and denominator.
  - If the degree of the numerator is smaller than the degree of the denominator, then there is a horizontal asymptote at $y = 0$
  - If the degree of the numerator is greater than the degree of the denominator, then there is no horizontal asymptote.
  - if the degree of the numerator is equal to the degree of the denominator, then there is a horizaontal asymptote and its height is given by the ratio of the leading terms.
  - For $f(x) = \dfrac{2(x-1)(x+3)}{(x+4)(x-1)}$, there is a horizontal asymptote at $y = 2$, since the ratio of leading terms (after you multiply out) is $\dfrac{2x^2}{x^2} = 2$
- The *x-intercepts* (also called *zeros*) are where $y = 0$, so that is where the numerator is zero.
  - For $f(x) = \dfrac{2(x-1)(x+3)}{(x+4)(x-1)}$, the only zero is at $x = 3$. $x = 1$ doesn't give an x-intercept because the $(x-1)$ factor cancels out (so there is a hole at $x = 1$ instead).

Warm-up problem 2: Without looking at the picture, find the vertical and horizontal asymptotes and the holes for the rational function

$$f(x) = \frac{(x+1)(x+3)(2x-3)}{(5x+1)(2x-3)(x-4)}$$



Problems:

1. Find the equation of a line with slope -5 through the point $(9, 1)$.

2. Find the equation of a line through the two points $(5, -1)$ and $(2, 4)$.

3. Where does the function $f(x) = \dfrac{x - 2}{2x^3 - 3x^2 - 2x}$ have:
   (a) holes
   (b) vertical asymptotes
   (c) horizontal asymptotes
   (d) zeros

Extra Problems

4. Find the equation of a line through the two points $(a, f(a))$ and $(a+h, f(a+h))$.

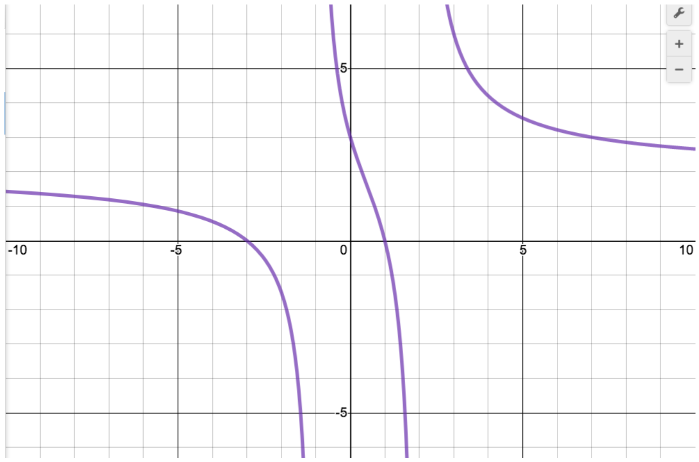
5. Find the horizontal asymptote(s), if any, for the folowing functions:

   (a) $f(x) = \dfrac{2x^3+1}{3x^3-6}$

   (b) $g(x) = \dfrac{x^2+1}{5x^3+2x^2+x-6}$

   (c) $h(x) = \dfrac{x^3}{2x+1}$

6. Find the equation for the rational function drawn:



ALEKS topics to work on relating to lines and graphs of rational functions

1. Finding slope given the graph of a line on a grid
2. Finding slope given two points on the line
3. Graphing a line through a given point with a given slope
4. Finding the slope and y-intercept of a line given its equation in the form Ax + By = C
5. Finding the slope, y-intercept, and equation for a linear function given a table of values
6. Writing an equation in point-slope form given the slope and a point
7. Writing an equation of a line given the y-intercept and another point
8. Writing the equation of the line through two given points
9. Writing the equations of vertical and horizontal lines through a given point
10. Finding slopes of lines parallel and perpendicular to a line given in the form Ax + By = C
11. Writing equations of lines parallel and perpendicular to a given line through a point
12. Graphing a rational function: Constant over linear
13. Graphing a rational function: Linear over linear
14. Graphing rational functions with holes

Videos on lines

- Calculus 1 Coreq Playlist > Lines: Graphs and Equations

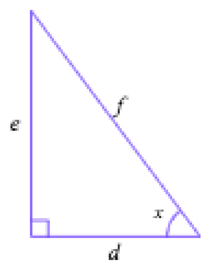
Videos on rational functions

- Calculus 1 Coreq Playlist > Rational Functions and Graphs

3 Trig Functions

In Calculus, you will soon learn the derivatives of trig functions, which can be described in terms of right triangles or the unit circle. Today will review trig function definitions and basic properties.

Warm-up Problem 1: A right triangle has sides as labeled.

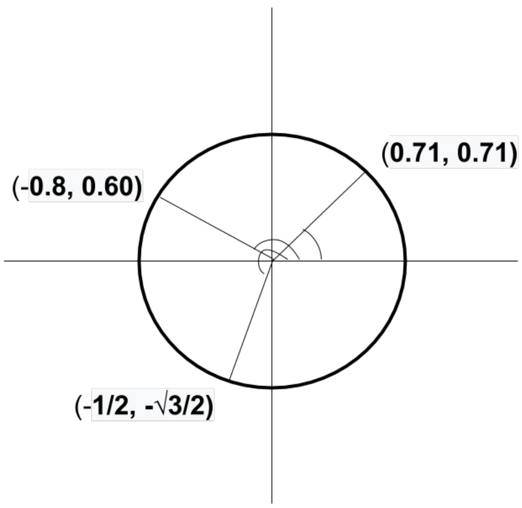


Find

1. $\sin(x)$
2. $\cos(x)$
3. $\tan(x)$
4. $\sec(x)$
5. $\csc(x)$
6. $\cot(x)$

Trig functions can also be evaluated via the unit circle. This is expecially handy for angles greater than 90°, that don't make sense for a right triange.

Warm-up Problem 2: Find $\sin(\theta)$ and $\cos(\theta)$ for the following angles on the unit circle, using the coordinates shown.

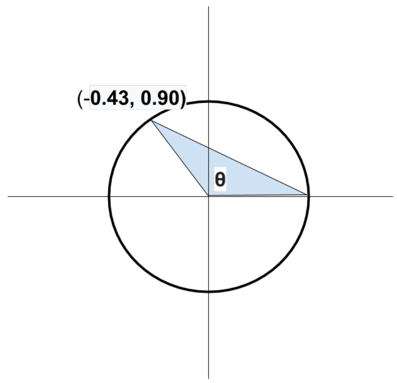


Warm-up Problem 3: If $\cos(\theta) = -\dfrac{3}{7}$ and θ is in the second quadrant, find

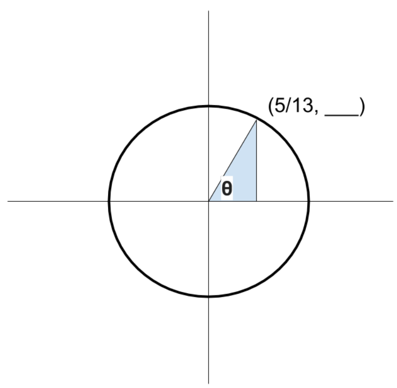
1. $\cos(-\theta)$
2. $\sin(-\theta)$
3. $\cos(\theta + 2\pi)$
4. $\sin(\theta)$
5. $\tan(\theta)$
6. $\tan(\theta + \pi)$

Problems:

1. For each figure, the triangle is inscribed in a unit circle, and one or both coordinates of its vertex is given. Find $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$.



(a)

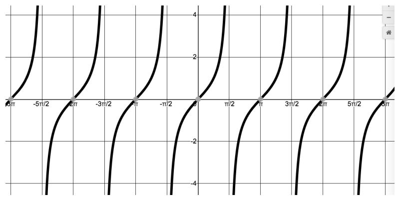


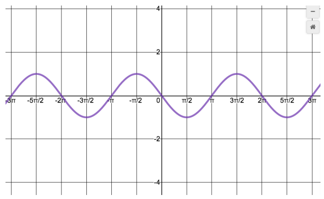
(b)

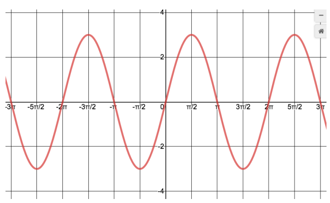
2. A 14 foot ladder leans against the side of a house. The angle of elevation of the ladder is 63°. How high is the top of the latter from the ground? Round your answer to the nearest tenth.

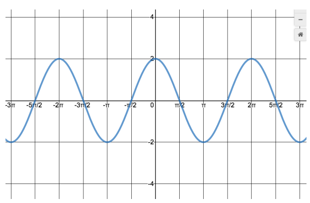
Extra Problems

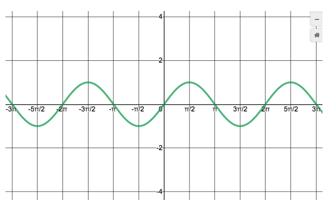
3. Match the graphs with the equations:
   - (i) $y = 3\sin(x)$
   - (ii) $y = 2\cos(x)$
   - (iii) $y = \cos(x - \pi/2)$
   - (iv) $y = -\sin(x)$
   - (v) $y = \tan(x)$

(a) 

(b) 

(c) 

(d) 

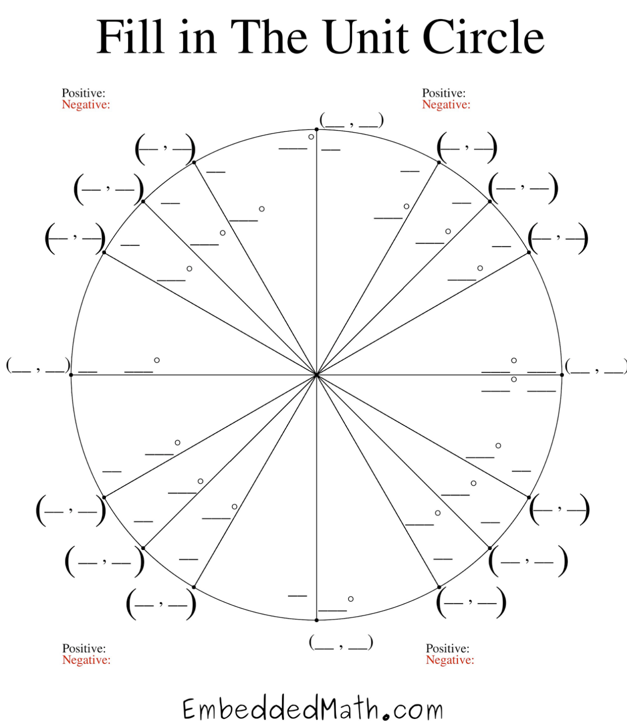
(e) 

4. The "special angles" that you need to memorize on the unit circle all have coordinates $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$ or $\frac{\sqrt{2}}{2}$ (or 0 or 1) with plus or minus signs.

   Place these numbers in the blanks on the first quadrant of this unit circle – WITHOUT looking it up!

   Hints:

   - the number $\frac{\sqrt{2}}{2}$ always goes with itself for both x and y coordinates.
   - the number $\frac{\sqrt{3}}{2}$ always goes with $\frac{1}{2}$
   - To decide which is the x-coordinate and which is the y-coordinate at a point, think about which number is bigger: $\frac{\sqrt{3}}{2}$ or $\frac{1}{2}$, and put the bigger number where it needs to go to match the geometry of the picture.



Now use symmetry and positive and negative signs to fill in the rest of the circle.

5. Let α be an angle in quadrant III such that $\sin(\alpha) = -\frac{7}{10}$ Find the exact values of $\sec(\alpha)$ and $\tan(\alpha)$.

ALEKS topics to work on relating to trig functions

1. Finding coordinates on the unit circle for special angles
2. Trigonometric functions and special angles: Problem type 1, 2, 3
3. Finding trigonometric ratios from a point on the unit circle
4. Even and odd properties of trigonometric functions
5. Sine, cosine, and tangent ratios: Variables for side lengths
6. Finding trigonometric ratios given a right triangle
7. Relationship between the sines and cosines of complementary angles
8. Using a trigonometric ratio to find a side length in a right triangle
9. Using trigonometry to find a length in a word problem with one right triangle
10. Solving a right triangle
11. Using trigonometry to find a length in a word problem with two right triangles
12. Finding values of trigonometric functions given information about an angle: Problem type 1, 2, 3, 4
13. Sketching the graph of y= a sin(x) or y= a cos(x)

Videos on Trig Functions

- Calculus 1 Coreq Playlist > Right Angle Trigonometry
- Calculus 1 Coreq Playlist > Sine and Cosine of Special Angles
- Calculus 1 Coreq Playlist > Unit Circle Definition of Sine and Cosine
- Calculus 1 Coreq Playlist > Properties of Trig Functions
- Calculus 1 Coreq Playlist > Properties of Trig Functions
- Calculus 1 Coreq Playlist > Graphs of Sine and Cosine
- Calculus 1 Coreq Playlist > Graphs of Sinusoidal Functions
- Calculus 1 Coreq Playlist > Graphs of Tan, Sec, Cot, Csc

4 Trig Identities

In order to rewrite trig expressions and solve trig equations, it's handy to know a few trig identities.

Here are the three most used trig formulas for Calc 1:

- $\cos^2(\theta) + \sin^2(\theta) = 1$ (Pythagorean identity)
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ (Double angle formula for sin)
- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ (Double angle formula for cos)

Sometimes it will be handy to rewrite trig functions in terms of other trig functions, so you'll need to know these definitions:

- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)} = \dfrac{1}{\tan(\theta)}$
- $\sec(\theta) = \dfrac{1}{\cos(\theta)}$
- $\csc(\theta) = \dfrac{1}{\sec(\theta)}$

Here are some other trig identites you may encounter less frequently in Calc 1:

- $\cos(-t) = \cos(t)$ (cosine is even)
- $\sin(-t) = -\sin(t)$ (sine is odd)
- $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$ (Angle sum for sin)
- $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(A)$ (Angle sum for cos)

There are many more trig identities, but most of the others can be found by putting a couple of the basic ones together. For example, if you need to rewrite $\cos(2\theta)$ just in terms of $\sin(\theta)$, you can first use the double angle formula

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

and then use

$$\cos^2(\theta) = 1 - \sin^2(\theta)$$

from the Pythagorean identity to rewrite further as

$$\cos(2\theta) = (1 - \sin^2(\theta) - \sin^2(\theta)$$

or

$$\cos(2\theta) = 1 - 2\sin^2(\theta)$$

Warm-up problem 1: Find all solutions to the equation

$$\cos(2x) + \sqrt{3}\cos(x) = -1$$

. Write your answer in radians in terms of π, using notation like: $x = \dfrac{\pi}{5} + 2k\pi, k \in Z$

Warm-up problem 2: Find all solutions to the equation $\tan(4x) = 1$.

Problems

1. Simplify the expressions:

   (a) $\sec(x)\cot(x)$

   (b) $\cos^3(x) + \cos(x)\sin^2(x)$

2. Find all solutions to the equation $2\sin(x) + \sin(2x) = 0$. Write your answer in radians in terms of π, using notation like: $x = \frac{\pi}{5} + 2k\pi, k \in Z$

3. Find all solutions to the equation $4\sin^2(3x) = 1$. in the interval $[0, 2\pi)$.

Extra Problem

4. Find all solutions of the equation in the interval $[0, 2\pi)$:

$$2\sin^2(\theta) - 3\sin(\theta) = 2$$

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ALEKS topics to work on relating to trig identities and equations

1. Simplifying trigonometric expressions
2. Double-angle identities: Problem type 1
3. Finding solutions in an interval for a basic equation involving sine or cosine
4. Solving a basic trigonometric equation involving sine or cosine
5. Finding solutions in an interval for a trigonometric equation in factored form
6. Finding solutions in an interval for a trigonometric equation with a squared function: Problem type 1, 2
7. Finding solutions in an interval for a trigonometric equation using Pythagorean identities: Problem type 1
8. Finding solutions in an interval for an equation with sine and cosine using double-angle identities
9. Solving a trigonometric equation involving a squared function: Problem type 1, 2
10. Solving a trigonometric equation involving an angle multiplied by a constant
11. Finding solutions in an interval for a trigonometric equation with an angle multiplied by a constant
12. Solving a trigonometric equation using double-angle identities

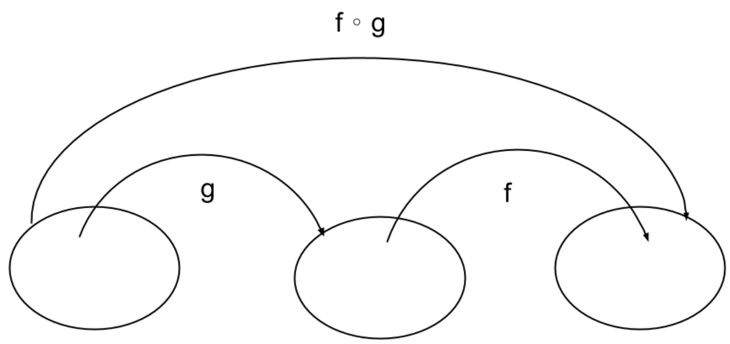
Videos on Trig Identities and Equations

- Calculus 1 Coreq Playlist > Solving Basic Trig Equations
- Calculus 1 Coreq Playlist > Trig Identities
- Calculus 1 Coreq Playlist > Pythagorean Identities
- Calculus 1 Coreq Playlist > Double Angle Formulas

5 Composition of Functions and Solving Rational Equations

In Calculus, you will soon learn the Chain Rule, which is about taking the derivative of a composition of functions.

For two functions $f(x)$ and $g(x)$, the composition $f \circ g(x)$ means $(f(g(x))$, which means you apply the function g first, and then apply f to the result.



Note that to evaluate a composition $f(g(x))$, you always work from the inside out, that is, you start with the innermost function.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$f(g(5)) = f(7) = (7)^2 = 49$$

and

$$f(g(x)) = f(x+2) = (x+2)^2$$

Be sure to use parentheses when plugging in a whole expression like $x + 2$ for x.

Warm-up Problem 1

Compute $f(g(x))$ where $f(x) = x^2 + x$ and $g(x) = \sqrt{x-3}$.

Warm-up Problem 2

Write $h(x)$ as a composition $h(x) = f(g(x))$ for some functions f and g.

$$h(x) = \frac{1}{5x-4}$$

Note: To write a function as a composition of two functions, it can be helpful to put a box around part of the expression for the function. Whatever is inside the box becomes your inner function, and whatever happens to the box gives your outside function.

In Calculus when doing implicit derivatives, you will have to solve equations for $\frac{dy}{dx}$.

The following steps, shown with an example, can help you solve an equation for the variable or expression you are interested in (e.g. $\frac{dy}{dx}$).

Example: Solve for y: $x = \frac{2-3y}{4+y}$

1. Clear the denominator by multiplying both sides by the least common denominator

$$(4+y)x = (4+y)\frac{2-3y}{4+y}$$

2. If the variable you want is trapped in parentheses, distribute to get rid of parentheses.

$$4x + yx = 2 - 3y$$

3. Move all the terms containing the variable you want to one side and all the terms that don't contain this variable to the other side.

$$yx + 3y = 2 - 4x$$

4. Factor out the variable you want and divide.

$$y(x+3) = 2 - 4x$$
$$y = \frac{2-4x}{x+3}$$

Problems

1. For each of the following functions, find functions $f(x)$ and $g(x)$ so that you can write the original function as a composition $f(g(x))$. Answers may vary.

   (a) $y = \sin\left(\frac{1}{x}\right)$

   (b) $y = \sin^2(x)$

2. Solve for y: $x + 2 = \frac{4+3y}{3-2y}$

3. Solve for $\frac{dy}{dx}$:

$$\frac{y + x\frac{dy}{dx}}{1 + yx} = x^2\frac{dy}{dx} + 2xy$$

Extra Problems

4. For each of the following functions, find functions $f(x)$ and $g(x)$ so that you can write the original function as a composition $f(g(x))$. Answers may vary.

   (a) $P(x) = \dfrac{1}{\sqrt{1+2x}}$

   (b) Can you write $P(x) = \dfrac{1}{\sqrt{1+2x}}$ as a composition of THREE functions $P(x) = f(g(h(x)))$

5. Solve for w: $\dfrac{3}{w-2} = \dfrac{-1}{w-5}$

ALEKS topics to work on relating to composition of functions

1. Composition of two functions: Basic and Advanced
2. Composition of two functions: Basic

ALEKS topics to work on relating to solving equations, especially rational equations

1. Solving a rational equation that simplifies to linear: Denominator x, or $x + a$, or a, x, or ax
2. Solving a rational equation that simplifies to linear: Unlike binomial denominators
3. Solving for a variable in terms of other variables in a rational equation: Problem type 1, 2, 3
4. Solving a rational equation that simplifies to quadratic: Binomial denominators, constant numerators
5. Solving a rational equation that simplifies to quadratic: Binomial denominators and numerators

Videos on composition of functions

- Calculus 1 Coreq Playlist > Composition of Functions

Videos on solving rational equations

- Calculus 1 Coreq Playlist > Solving Rational Equations

6 Logarithms

In Calculus, you will finding derivatives of log and exponential functions and using a method called logarithmic differentiation to calculate derivatives. This section reviews properties of logarithms and their graphs.

Recall that $\log_a b = c$ means that $a^c = b$. In both expressions, a is the base.

You'll need to be comfortable with the following log properties, that are the equivalents of the exponent rules.

- Product rule: $\log_a(xy) \log_a(x) + \log_a(y)$
- Quotient rule: $\log_a (xy) = \log_a(x) - \log_a(y)$
- Power rule: $\log_a x^m = m \log_a(x)$
- Inverse rules: $\log_a a^x = x$ and $a^{\log_a x} = x$

Warm-up Example 1

Simplify the following expressions, if possible.

1. $\log_4 64$
2. $\log_3 \dfrac{1}{81}$
3. $\log_5 1$
4. $\log_2 0$
5. $\ln \sqrt{e}$
6. $2^{\log_2 y}$
7. $e^{-\ln x}$
8. $\dfrac{\log(x^3)}{\log(\sqrt{x})}$
9. $\ln(x + 1)$

Warm-up Example 2

Graph the function and give its domain and range using interval notation.

$y = \log_2(x + 2)$

Warm-up Example 3

Use log properties to expand the expression. Each logarithm should involve only one variable and should not have any exponents. Assume that all variables are positive.

$\ln \frac{x^4 \cdot x^x}{(x+5)^3}$

Problems

1. Simplify the following expressions:

   (a) $\log_6 \frac{1}{6}$

   (b) $\ln e^x + \ln e$

   (c) $3^{x \ln 3}$

   (d) $\ln\left(\frac{1}{x}\right) + \ln(x)$

   (e) $\frac{\ln(e^2 x^4)}{\ln e^2}$

2. Use the properties of logarithms to expand the expression. Each logarithm should involve only one variable and should not have any exponents. Assume that all variables are positive.

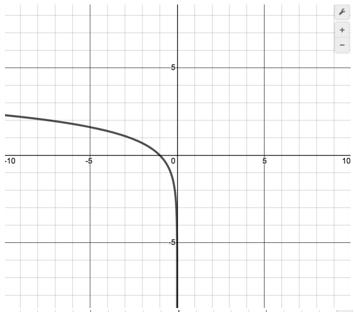
   (a) $\log \dfrac{x^5(x-2)^2}{(x+5)^4}$

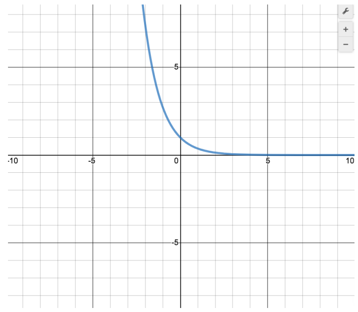
   (b) $\ln\left(5x^{\cos x}\right)$

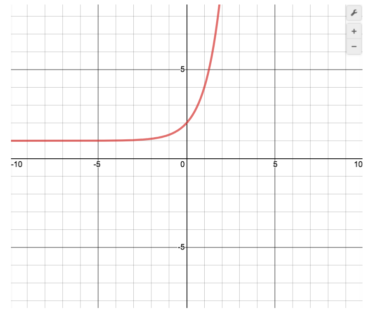
3. Write the expression as a single log.

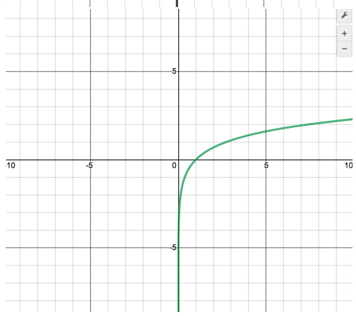
   $5\log_b x - \dfrac{1}{4}\log_b w + 6\log_b y$

4. Match the graphs with the equations.

1. 

2. 

3. 

4. 

(a) $y = 3^x + 1$

(b) $y = e^{-x}$

(c) $y = \ln(x)$

(d) $y = \ln(-x)$

ALEKS topics relating to logarithms and exponential functions

1. Graphing an exponential function and its asymptote: $f(x) = ab^x$
2. Translating the graph of an exponential function
3. The graph, domain, and range of an exponential function
4. Converting between logarithmic and exponential equations
5. Converting between natural logarithmic and exponential equations
6. Evaluating logarithmic expressions
7. Solving an equation of the form $\log_b a = c$
8. Translating the graph of a logarithmic function
9. Graphing a logarithmic function: Basic
10. The graph, domain, and range of a logarithmic function
11. Domain of a logarithmic function: Advanced
12. Basic properties of logarithms
13. Expanding a logarithmic expression: Problem type 1
14. Writing an expression as a single logarithm
15. Solving a multi-step equation involving a single logarithm: Problem type 2
16. Solving a multi-step equation involving natural logarithms
17. Solving an equation involving logarithms on both sides: Problem type 2
18. Solving an exponential equation by using natural logarithms: Decimal answers
19. Solving an exponential equation by using logarithms: Exact answers in logarithmic form

Videos on Logarithms

- Calculus 1 Coreq Playlist > Logarithms: Introduction
- Calculus 1 Coreq Playlist > Log Functions and Graphs
- Calculus 1 Coreq Playlist > Combining Logs and Exponents
- Calculus 1 Coreq Playlist > Log Rules

7 Inverse Functions, Inverse Trig Functions

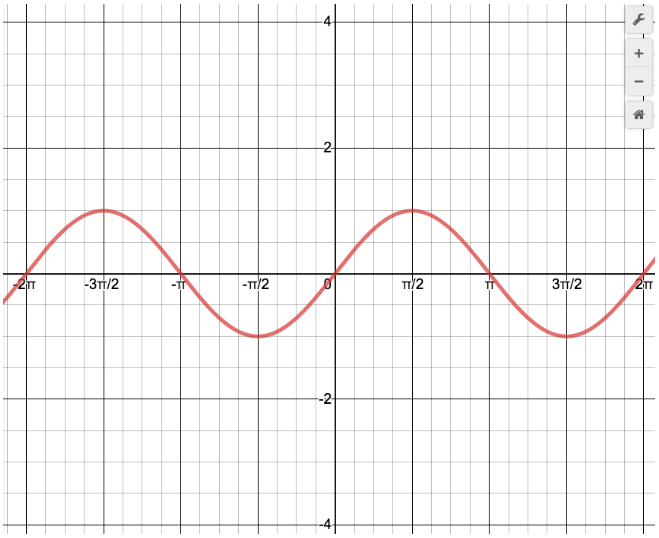
This section gives a review of inverse trig functions in preparation for doing calculus with them, e.g. taking their derivatives.

Recall that the inverse of a function f is the function f^{-1} that undoes what the function does. For example, if $f(x) = x^3$, then $f^{-1}(x) = \sqrt[3]{x}$. Here are some properties of inverse functions in general:

1. Inverse functions reverse the roles of y and x. You can find the inverse of a function by reversing y and x and then solving for y.
2. The graph of $y = f^{-1}(x)$ is obtained from the graph of $y = f(x)$ by reflecting over the line $y = x$.
3. $f^{-1} \circ f(x) = x$ and $f \circ f^{-1}(x) = x$. This is the mathematical way of saying that f and f^{-1} undo each other.
4. A function f has an inverse function if and only if the graph of f satisfies the **horizontal line test** (i.e. every horizontal line intersects the graph of $y = f(x)$ in at most one point.)
5. For any invertible function f, the domain of $f^{-1}(x)$ is the same as the range of $f(x)$ and the range of $f^{-1}(x)$ is the same as the domain of $f(x)$.

Warm-up Problem 1:

Consider the function $y = \sin(x)$



1. Does it have an inverse function?
2. What interval of x-values could we restrict the function to, so that it will have an inverse function?
3. With that restriction, graph the inverse function $y = \sin^{-1}(x)$ on the same axis.
4. What are the domain and range of $\sin^{-1}(x)$?
5. Evaluate
   (a) $\sin^{-1}(0.5)$
   (b) $\arcsin(-\frac{\sqrt{3}}{2})$. (Note arcsin means the same thing as $\sin^{-1}(x)$.
6. True or False: $\sin^{-1}(x) = \dfrac{1}{\sin(x)}$

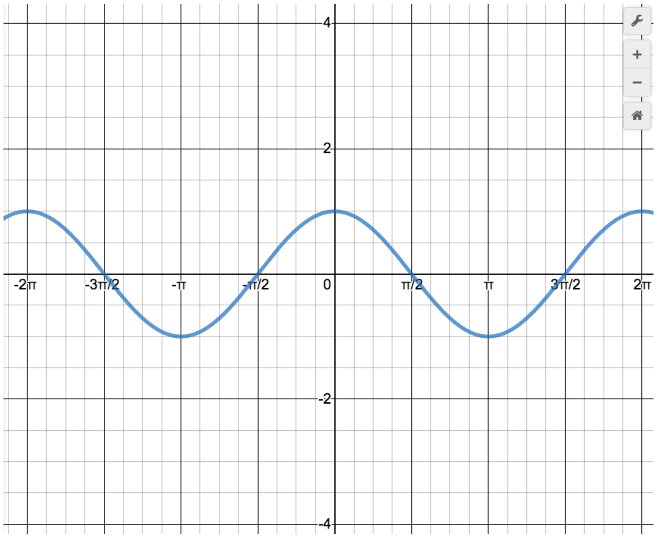
Warm-up Problem 2:

Evaluate $\cos(\sin^{-1}(\frac{4}{5}))$

Hint: use a right triangle.

Inverse Cosine: (see video)

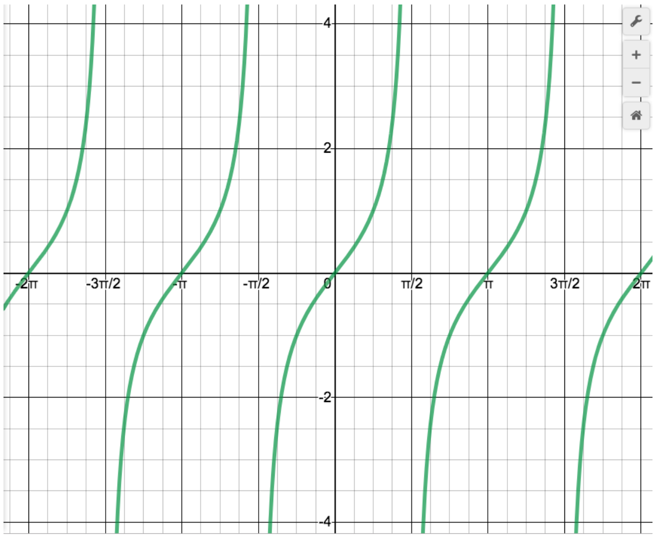
Consider the function $y = \cos(x)$



1. Does it have an inverse function?
2. What interval of x-values could we restrict the function to, so that it will have an inverse function?
3. With that restriction, graph the inverse function $y = \cos^{-1}(x)$ on the same axis.
4. What are the domain and range of $\cos^{-1}(x)$?
5. Evaluate
   (a) $\cos^{-1}(-\frac{\sqrt{3}}{2})$
   (b) $\arccos(0)$
6. True or False: $\cos^{-1}(x) = \dfrac{1}{\cos(x)}$

Inverse Tan: (see video)

Consider the function $y = \tan(x)$



1. Does it have an inverse function?
2. What interval of x-values could we restrict the function to, so that it will have an inverse function?
3. With that restriction, graph the inverse function $y = \tan^{-1}(x)$ on the same axis.
4. What are the domain and range of $\tan^{-1}(x)$?
5. Evaluate
   (a) $\tan^{-1}(1)$
   (b) $\arctan(-\sqrt{3})$
6. What is $\lim_{x\to\infty} \arctan(x)$?

Problems:

1. Evaluate:

   (a) $\arcsin(-\frac{1}{2})$

   (b) $\tan^{-1}(0)$

2. Evaluate, if possible:

   (a) $\sin(\arcsin(\frac{\sqrt{3}}{2}))$

   (b) $\sin^{-1}(\sin(\frac{-4\pi}{5}))$

3. Evaluate $\cos(\tan^{-1}(\frac{8}{15}))$. Hint: use a right triangle.

4. Rewrite as an algebraic expression in w.

   $\sec(\sin^{-1}(6w))$

Extra Problems:

5. Evaluate:
   (a) $\arccos(1)$
   (b) $\cos^{-1}(\frac{\sqrt{3}}{2})$
6. Evaluate, if possible:
   (a) $\sin(\sin^{-1}(-2))$
   (b) $\arccos(\cos(\frac{8\pi}{5}))$
7. Evaluate $\cos(\sin^{-1}(-\frac{7}{25}))$. Hint: use a right triangle.
8. Rewrite as an algebraic expression in w. $\tan(\cos^{-1}(\frac{w}{6}))$

ALEKS topics relating to inverse trig functions

1. Values of inverse trigonometric functions
2. Composition of a trigonometric function with its inverse trigonometric function: Problem typea 1, 2
3. Composition of a trigonometric function with the inverse of another trigonometric function: Problem type 1
4. Composition of trigonometric functions with variable expressions as inputs: Problem type 1, 2
5. Solving a right triangle

Videos on inverse functions and inverse trig functions

- Calculus 1 Coreq Playlist > Inverse Functions
- Calculus 1 Coreq Playlist > Inverse Trig Functions
- Calculus 1 Coreq Playlist > Solving Right Triangles

8 Related Rates Set-up

Related rates problems generally involve at least 3 different variables; for example, x, y, and t. If you can find an equation relating x and y, then by taking the derivative of both sides, you can deduce a relationship between the derivatives $\frac{dy}{dt}$ and $\frac{dx}{dt}$, so the derivatives, or "rates" are "related". This allows you to find $\frac{dy}{dt}$, for example, only given $\frac{dx}{dt}$ as well as x.

Here are some problems on setting up equations, that are good practice for setting up related rates problems.

Warm-up Problem 1:

A baseball diamond is a square whose sides are 90 ft long. Suppose that a player running from second base to third base has a speed of 30 ft/s at the instant when he is 20 ft from third base. At what rate is the player's distance from home plate changing at that instant?

Write down an equation to relate the variables in this problem. You do not need to solve the problem.

Related Rates Set-up Problems

For each problem, write down an equation to relate the variables in this problem. You do not need to solve the problems.

1. Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of 2 ft/s. How fast is the area of the spill increasing when the radius of the spill is 60 ft?

2. A rocket is rising vertically at 880 ft/s. A camera that is 3000 ft away is tracking its motion. How fast must the camera elevation angle change at that instant to keep the rocket in sight, at the moment when the rocket is 4000 ft high?

3. A liquid is to be cleared of sediment by pouring it through a conical filter that is 16 cm high and has a radius of 4 cm at the top. Suppose that the liquid flows out at 2 cm^3/min. At what rate is the depth of the liquid changing at the instant when the level is 8 cm deep?

ALEKS topics related to setting up equations

1. Area of a piecewise rectangular figure
2. Area of a triangle
3. Perimeter involving rectangles and circles
4. Circumference and area of a circle: Exact answers in terms of pi
5. Area involving inscribed figures
6. Volume of a rectangular prism
7. Volume of a cylinder
8. Surface area of a cube or a rectangular prism
9. Surface area of a cylinder: Exact answers in terms of pi
10. Word problem involving the Pythagorean Theorem
11. Finding the perimeter or area of a rectangle given one of these values
12. Similar right triangles
13. Indirect measurement
14. Using trigonometry to find a length in a word problem with one right triangle
15. Using trigonometry to find angles of elevation or depression in a word problem
16. Using trigonometry to find a length in a word problem with two right triangles

Videos on setting up equations

None yet, although there are a few videos that set up and solve related rate problems in the Calculus playlist

- Calculus 1 Playlist > Related Rates - Distances
- Calculus 1 Playlist > Related Rates - Volume and Flow
- Calculus 1 Playlist > Related Rates - Angle and Rotation

9 Inequalities, Intercepts, and Domains

In Calculus, you will be graphing functions, using derivatives and sign charts. For example, you'll need to know where the derivative is positive to see where the function is increasing. The following problems solving inequalities will build your skills using sign charts, as well as some graphing tools like finding intercepts.

Warm-up Problem 1: Solve the inequality

$$\frac{4x-12}{x^3+2x^2+x} > 0$$

.

Warm-up Problem 2: Find the domain, x-intercepts, and y-intercepts of $f(x) = \dfrac{\sqrt{x^2 - 4}}{x + 5}$.

1. Solve the inequalities:

   (a) $4x^3 - 12x^2 + 8x < 0$

   (b) $\dfrac{8(x^2-1)}{3\sqrt[3]{x}} < 0$

2. Find the domain, x-intercepts, and y-intercept of the function

   (a) $f(x) = \dfrac{2x-3}{2x-8}$

(b) $f(x) = \ln(x^2 - 1)$

Extra Problems

3. Solve the inequality $\dfrac{e^x - e^{3x}}{(1 + e^{2x})^2} > 0$

4. Find the domain, x-intercepts, and y-intercept of the function $f(x) = x - 3x^{2/3}$

Problems in ALEKS related to inequalities, intercepts, and domain.

1. Finding intercepts of a nonlinear function given its graph
2. Domain of a rational function: Excluded values
3. Domain of a square root function: Advanced
4. Solving a quadratic inequality written in factored form
5. Solving a polynomial inequality: Problem type 1, 2, 3, 4
6. Solving a rational inequality: Problem type 1

Videos on inequalities

- Calculus 1 Coreq Playlist > Polynomial and Rational Inequalities

10 Setting up Equations for Optimization Problems

Here are some general tips for setting up and solving optimization problems.

1. Find the function $f(x)$ which you are trying to optimize (i.e. find the minimum or maximum value of). This function should be in terms of an independent variable x.

   For example, in the first warm up problem below, the quantity to be optimized is distance travelled (which should be minimized). The independent variable here could be, for example, the angle that the boat makes with the horizontal. From A to C directly is zero degrees.

2. Use constraints on the problem to make sure that the function you are trying to optimize is in terms of only one variable. A "helper" equation may be necessary here.

3. Figure out the domain interval of interest.

   For example, in the problem below we can only consider angles between 0 and 90 degrees. The final answer can only lie in the domain interval of interest.

4. Set $f'(x) = 0$ in order to find all the critical points x in the domain of interest where $f'(x)$ is defined. Also consider points x in the domain of interest where $f'(x)$ is not defined.

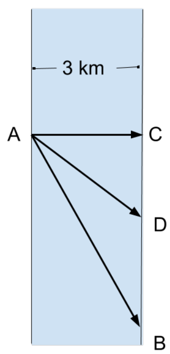
5. If the domain of interest is a closed interval, consider the endpoints. The max or min value of the function could occur at an endpoint.

6. Compute $f(x)$ for each candidate point x from the previous two steps, and figure out what the max (or min) value is.

Often, the hardest part of an optimization problem is setting up the equation(s), which is what we will focus on here.

Warm-up Problem 1:

A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (Assume the speed of the water is negligible.)



Warm-up Problem 2: We need to enclose a rectangular field with a fence. We have 500 feet of fencing material and a building is on one side of the field and so won't need any fencing. Determine the dimensions of the field that will enclose the largest area.

Optimization Set-up Problems

For each problem, write down an equation for the quantity that needs to be maximized or minimized. You may need to use a "helper" equation to eliminate variables until you are down to one variable. You do not need to solve the problems.

1. An open box is formed from a square sheet of cardboard by cutting equal squares from each corner and folding up the edges. If the dimensions of the cardboard are 18 cm by 18 cm, what should be the dimensions of the box so as to maximize the volume, and what is the maximum volume?

2. We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost 10 dollars per square foot, and the material used to build the sides costs 6 dollars per square foot. If the box must have a volume of 50 cubic feet, determine the dimensions that will minimize the cost to build the box.

3. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest from the point $(1, 0)$. Hint: the "helper" equation is $4x^2 + y^2 = 4$ and you will need to use the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Extra Problems

4. A farmer wishes to enclose a 4000 square meter field and subdivide the field into four rectangular plots with fences parallel to one of the sides. What should be the dimensions of the field be in order to minimize the amount of fencing required?

5. Suppose a rectangle has its lower base on the x-axis and upper vertices on the graph of the function $y = 8 - x^2$. Find the area of the largest such rectangle.

6. A rectangular storage container with an open top is to have a volume of 10 m^2. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

7. A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

8. A motorist is in the desert in a Jeep. The closest point on a straight road is town A and it is $4\sqrt{2}$ miles from the motorist. He wishes to reach town B, 10 miles from town A on the road, in the shortest time. If he can drive 15 mi/hr on the desert and 45 mi/hr on the road, where should he intersect the road?

ALEKS topics related to setting up equations

1. Area of a piecewise rectangular figure
2. Area of a triangle
3. Perimeter involving rectangles and circles
4. Circumference and area of a circle: Exact answers in terms of pi
5. Area involving inscribed figures
6. Volume of a rectangular prism
7. Volume of a cylinder
8. Surface area of a cube or a rectangular prism
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12. Similar right triangles
13. Indirect measurement
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16. Using trigonometry to find a length in a word problem with two right triangles

Videos on setting up optimization problems

None yet.

11 More Practice on Related Rates and Optimization Problems

Warm-up Problem 1: The manager of a 100-unit apartment complex knows from experience that all units will be occupied if the rent is \$800 per month. A market survey suggests that, on average, one additional unit will remain vacant for each \$10 increase n rent. What rent should the manager charge to maximize revenue?

Warm-up Problem 2: A two-piece extension ladder leaning against a wall is collapsing at a rate of 2 ft/sec. while the foot of the ladder remains a constant 5 ft from the wall. How fast is the ladder moving down the wall when the ladder is 13 ft. long?

Problems:

1. Suppose that you are standing on the ground, under the flight path of an airplane. The airplane is flying at an altitude of 10 km. Using a radar device, you are able to detect that the plane is currently 26 km away from you, and that its distance to you is currently shrinking at a rate of 840 kilometers per hour. What is the current speed of the airplane?

2. Which point on the graph of $y = \sqrt{x}$ is closest to the point $(7, 0)$?

3. A cone is constrained to have a volume of 100 cubic feet. Find the height and radius that minimize the surface area of the cone. Hint: the surface area of a cone is given by the formula: $A = \pi r^2 + \pi r s$, where s is the "slant height" given by $s^2 = r^2 + h^2$, and h is the height.

4. There is a hole at the tip of a cone filled with water, and it starts leaking at a rate of 3 cubic feet per minute. The cone has a height of 10 feet with a diameter of 8 feet. Determine how fast the height of the water is falling when the water only reaches up to half the height of the cone. Hint: the diameter is twice the radius, and the volume of a cone is: $V = \dfrac{1}{3}\pi r^2 h$.

Extra Problems:

5. A 6 foot man walks away from a 12 foot lamp post at the rate of 4 ft/sec. How fast is his shadow lengthening when he is 21 feet from the post?

6. A poster is to have an area of 180 cm^2 with 1 cm margins at the bottom and sides and a 2 cm margin at the top. What dimensions will give the largest printed area?

7. Gas is escaping from a spherical balloon at a rate of 2 ft3/min. How fast is the surface area shrinking when the radius is 12ft?

8. A rectangular field is to be enclosed by a fence and divide into 3 lots by fences parallel to one of its sides. Find the dimensions of the largest field that can be enclosed with a total of 800 meters of fencing.

9. An open box is formed from a rectangular sheet of cardboard by cutting equal squares from each corner and folding up the edges. If the dimensions of the cardboard are 15 in by 24 in, what size squares should be cut to obtain a box of maximum volume?

10. Two cars leave an intersection. One travels north at 30mph, the other travels east at 40mph. How fast is the distance between them increasing at the end of 30 minutes?

11. Find the base radius r and the height h of the right circular cone of maximum volume that will fit inside a sphere of radius 3 units.

12 Review of Limits and Derivatives

Problems:

1. Evaluate:

$$\lim_{h \to 0} \frac{(6+h)^2 - 36}{h}$$

   if it exists. If not, explain why not.

2. Compute the following derivatives:

   (a) $f(x) = \dfrac{x}{e^x - 1}$

   (b) $g(x) = 2\sin(3x + \tan(x))$

3. Find $\dfrac{dy}{dx}$ for the curve given by the equation

$$7y^2 + \sin(3x) = 12 - y^4$$