§14.1 Vector Functions

After completing this section, students should be able to:

- Define a vector valued function.
- Find the domain of a vector valued function.
- Find the limit of a vector valued function.
- Match equations of vector valued functions with their graphs by considering the projections of the graphs onto the xy, yz, and xz planes.
- Give a vector valued equation for the intersection of two surfaces.

Definition. A **vector function** or **vector-valued function** is:

If we think of the vectors as position vectors with their initial points at the origin, then the endpoints of $\vec{v}(t)$ trace out a ________ in $\mathbb{R}^3$ (or in $\mathbb{R}^2$).

Example. Sketch the curve defined by the vector function $\vec{r}(t) =< t, \sin(5t), \cos(5t) >$.

Example. Consider the vector function $\vec{r}(t) = \dfrac{t^2 - t}{t - 1}\vec{i} + \sqrt{t + 8}\,\vec{j} + \dfrac{\sin(\pi t)}{\ln t}\vec{k}$

1. What is the domain of $\vec{r}(t)$?

2. Find $\lim_{t \to 1} \vec{r}(t)$

3. Is $\vec{r}(t)$ continuous on $(0, \infty)$? Why or why not?

END OF VIDEO

Review. Which of these are vector functions?

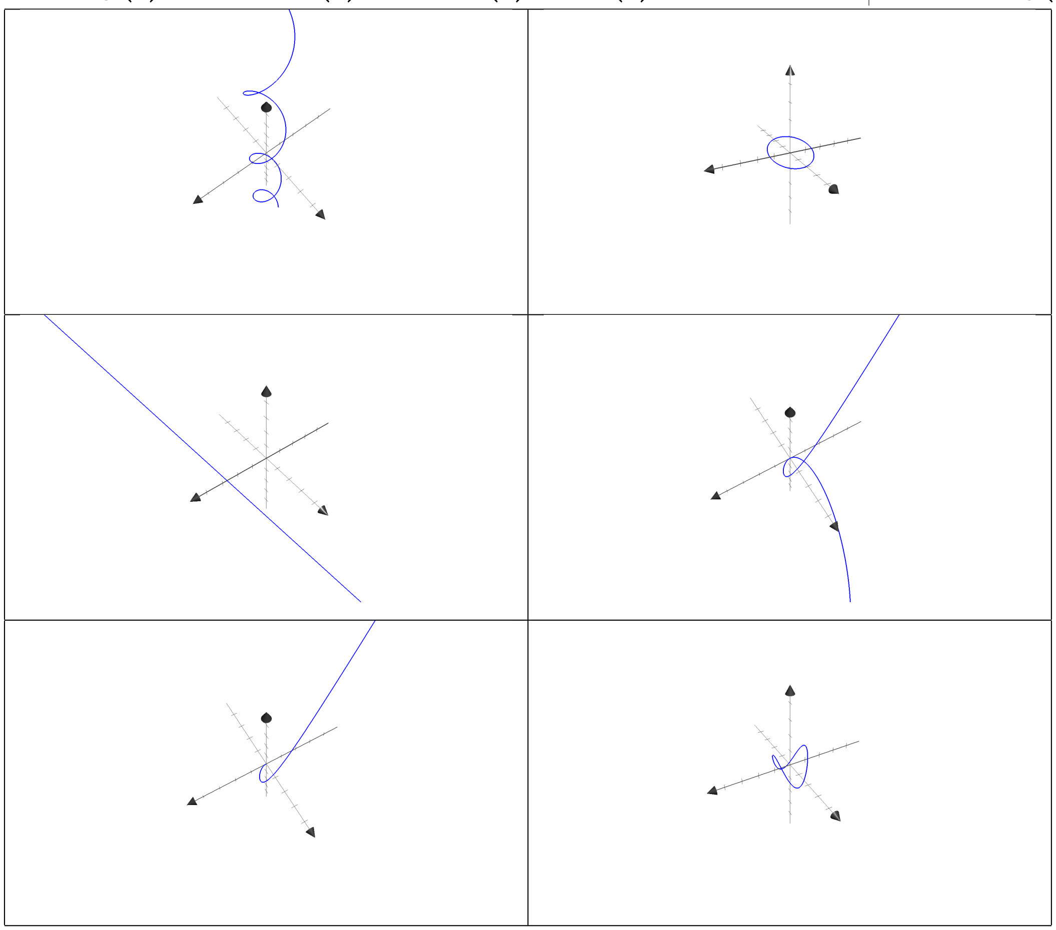
A. $f(t)) = t^2$

B. $f(s,t) = 3x - 4t$

C. $f(t) = t^2\vec{i} - 2t\vec{j} + \sqrt{t}\vec{k}$

Review. Match the vector functions with the curves.

1. $\vec{r_1}(t) =< t^2, t^4, t^6 >$
2. $\vec{r_2}(t) =< t + 2, 3 - t, 2t - 1 >$
3. $\vec{r_3}(t) =< \cos(t), -\cos(t), \sin(t) >$
4. $\vec{r_4}(t) =< t, t^2, t^3 >$
5. $\vec{r_5}(t) =< \cos(t), \sin(t), t >$
6. $\vec{r_6}(t) =< \cos(t), \sin(t), \cos(2t) >$



Example. Consider the vector function $\vec{r}(t) = te^{-t}\vec{i} + \dfrac{t^3 + t}{2t^3 - 1}\vec{j} + \dfrac{1}{\sqrt{t}}\vec{k}$

1. What is the domain of $\vec{r}(t)$?

2. Find $\lim_{t \to \infty} \vec{r}(t)$

Example. Find the point on the curve $\vec{r}(t) = 5\cos(t)\vec{i} + 3\sin(t)\vec{j} + 4\sin(t)\vec{k}$ that lies closest to the point $P(1, 1, 2)$.

Example. At what points does the helix $\vec{r}(t) =< \sin(t), \cos(t), t >$ intersect the sphere $x^2 + y^2 + z^2 = 5$?

Example. Show that the curve $\vec{r}(t) = 3\cos(t)\vec{i} + 9\cos(2t)\vec{j} + 3\sin(t)\vec{k}$ lies on the intersection of the hyperboloid $y = x^2 - z^2$ and the cylinder $x^2 + z^2 = 9$.

Extra Example. Find a function $\vec{r}(t)$ that describes the curve where the following surfaces intersect.

$z = 3x^2 + y^2 + 1$, $z = 5 - x^2 - 3y^2$

Extra Example. Find the curve where the following surfaces intersect.

$x^2 + y^2 = 25, z = 2x + 2y$

Extra Example. Find the curve where the following surfaces intersect.

$z = y + 1, z = x^2 + 1$

Extra Example. §14.2 Derivatives and Integrals of Vector Functions

By the end of this section, students should be able to:

- Compute the derivative of a vector function.
- Compute the integral of a vector function.
- When $\vec{r}(t)$ represents the position of a particle at time t, explain the meaning of $\vec{r}'(t)$, its direction, and its magnitude.

Suppose a particle is moving according to the vector equation $\vec{r}(t)$. How can we find a *tangent vector* that gives the direction and speed that the particle is traveling?

Definition. The *derivative* of the vector function $\vec{r}(t)$ is the same thing as the tangent vector, defined as

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) =$$

If $\vec{r}(t) =< r_1(t), r_2(t), r_3(t) >$, then

$$\vec{r}\,'(t) =$$

The derivative of a vector function is a (circle one) vector / scalar.

The **unit tangent vector** is:

$\vec{T}(t) =$

The **tangent line** at $t = a$ is:

Example. For the vector function $\vec{r}(t) =< t^2, t^3 >$

1. Find $\vec{r}\,'(1)$.
2. Sketch $\vec{r}(t)$ and $\vec{r}\,'(1)$.
3. Find $\vec{T}(1)$.
4. Find the equation for the tangent line at $t = 1$.

Definition. If $\vec{r}(t) =< f(t), g(t), h(t) >$, then

$$\int \vec{r}(t)\, dt =$$

and

$$\int_a^b \vec{r}(t)\, dt =$$

Example. Compute $\int_1^2 \frac{1}{t}\vec{i} + e^t\vec{j} + te^t\vec{k}$.

END OF VIDEO

Review. If $\vec{r}(t) = 5t^2\vec{i} + \sin(t)\vec{j} - 3\vec{k}$, how do we compute $\vec{r}\,'(t)$?

Question. For a vector function $\vec{r}(t) =< r_1(t), r_2(t), r_3(t) >$, what is the limit definition of $\vec{r}\,'(t)$?

Question. (Geometric interpretation) If we think of $\vec{r}(t)$ as a space curve, what does $\vec{r}\,'(t)$ represent geometrically?

Question. (Physics interpretation) If $\vec{r}(t)$ represents the position of a particle at time t,

(a) what does the direction of $\vec{r}\,'(t)$ signify?

(b) what does the magnitude of $\vec{r}\,'(t)$ signify?

Note. The *unit tangent vector* is computed as ________________ and sometimes denoted by ________________ .

Example. Find the tangent vector, the unit tangent vector, and the tangent line for the following curves at the point given

1. $\vec{r}(t) =< t, t^2, t^3 >$ at $t = 1$

2. $\vec{r}(t) =< t^2, t^4, t^6 >$ at $t = 1$

3. $\vec{p}(t) =< t + 2, 3 - t, 2t - 1 >$ at $t = 0$

Example. At what point do the curves $\vec{r_1}(t) =< t, 1-t, 3+t^2 >$ and $\vec{r_2}(t) =< 3-t, t-2, t^2 >$ intersect? Find their angle of intersection correct to the nearest degree.

Derivative rules - see textbook

- Is there a product rule for derivatives of vector functions?
- Is there a quotient rule for derivatives of vector functions?
- Is there a chain rule for derivatives of vector functions?

Example. Show that if $\|\vec{r}(t)\| = c$ (a constant), then $\vec{r}\,'(t)$ is orthogonal to $\vec{r}(t)$ for all t.

Review. If $\vec{r}(t) =< f(t), g(t), h(t) >$, then

$$\int \vec{r}(t)\, dt =$$

and

$$\int_a^b \vec{r}(t)\, dt =$$

Example. Find $\vec{p}(t)$ if $\vec{p}\,'(t) = \cos(\pi t)\vec{i} + \sin(\pi t)\vec{j} + t\vec{k}$ and $\vec{p}(1) = 6\vec{i} + 6\vec{j} + 6\vec{k}$.

Extra Example. Show that if $\vec{r}$ is a vector function such that $\vec{r}\,''$ exists, then

$$\frac{d}{dt}[\vec{r}(t) \times \vec{r}\,'(t)] = \vec{r}(t) \times \vec{r}\,''(t)$$

Extra Example. If $\vec{u}(t) = \vec{r}(t) \circ [\vec{r}\,'(t) \times \vec{r}\,''(t)]$, show that

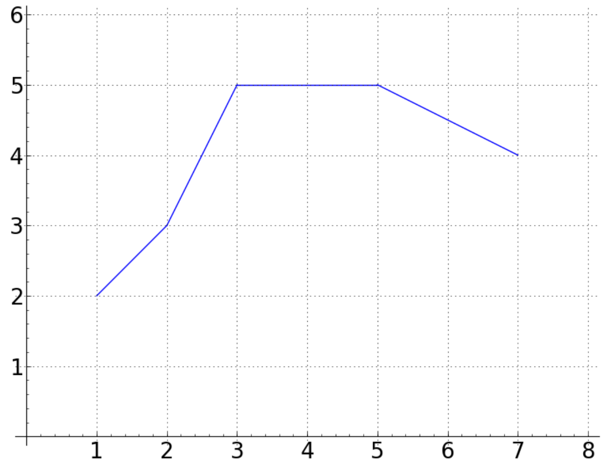
$$\vec{u}\,'(t) = \vec{r}(t) \cdot [\vec{r}\,'(t) \times \vec{r}\,'''(t)]$$

§14.4 Arclength

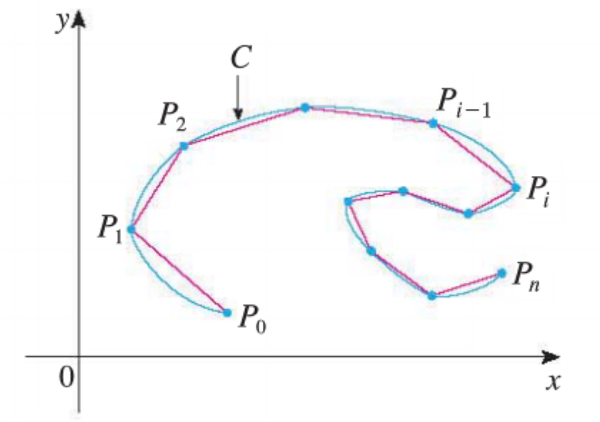
After completing this section, students should be able to:

- Set up an intergral to represent the arclength of a curve, and compute the integral when it simplifies nicely.
- Explain what it means for a curve to be parametrized by arclength.
- Reparametrize curves so that they are parametrized by arclength.

Example. Find the length of this curve.



Note. In general, it is possible to approximate the length of a curve $x = f(t)$, $y = g(t)$ between $t = a$ and $t = b$ by dividing it up into n small pieces and approximating each curved piece with a line segment.

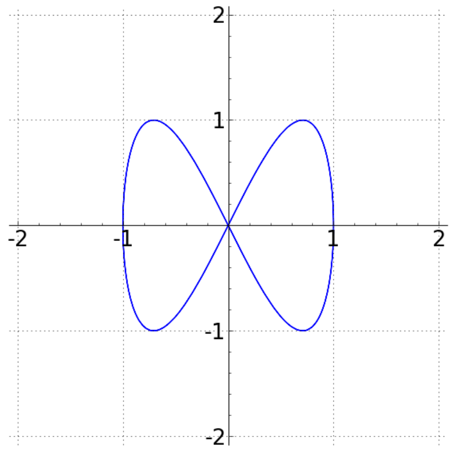


Arc length is given by the formula:

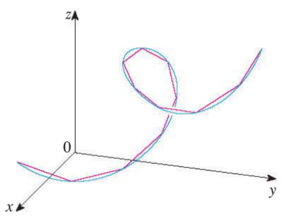
Set up an integral to express the arclength of the Lissajous figure

$$x = \cos(t), y = \sin(2t)$$

.



Review. To find the arc length of a space curve $\vec{r}(t) =< f(t), g(t), h(t) >$, we can approximate it with straight line segments.



Note. The **arc length** of a curve $\vec{r}(t) =< x(t), y(t), z(t) >$ between $t = a$ and $t = b$ is given by

Definition. The **arc length function** (starting at $t = a$) is

$$s(t) =$$

Note. If $s(t)$ is the arc length function, then $s'(t) =$

In words, this says that the rate of change of the arclength with respect to time is ...

Example. Consider the two curves:

1. $\vec{r}(u) =< 2u, u^2, \frac{1}{3}u^3 >$ for $0 \leq u \leq 1$
2. $\vec{q}(t) =< 2\ln(t), (\ln(t))^2, \frac{1}{3}(\ln(t))^3 >$ for $1 \leq t \leq e$

How are the curves related?

We say that $\vec{q}(t)$ is a **reparametrization** of $\vec{r}(u)$ because:

Also $\vec{r}(u)$ is a reparametrization of $\vec{q}(t)$ because:

You can think of a reparametrization of a curve as the same curve, traveled at a different speed. In our case, $\vec{q}$ moves along the curve (circle one) slower / faster than $\vec{r}$.

In mathematical notation, $\vec{q}(t)$ is a reparametrization of $\vec{r}(u)$ if $\vec{q}(t) = \vec{r}(\phi(t))$ for some strictly increasing (and therefore invertible) function $u = \phi(t)$.

Find the arc length of each curve.

$\vec{r}(u) =< 2u, u^2, \frac{1}{3}u^3 >$
for $0 \leq u \leq 1$

$\vec{q}(t) =< 2\ln(t), (\ln(t))^2, \frac{1}{3}(\ln(t))^3 >$
for $1 \leq t \leq e$

Fact. Arc length does not depend on parametrization.

Proof:

Is there a natural, best way to parametrize a curve?

Definition. We say that a curve $\vec{r}(t)$ is parametrized by arclength if ....

Note. If the curve $\vec{r}(t)$ is parametrized by arclength then ...

Note. If $\|\vec{r}\,'(t)\| = 1$ for all t, then ...

Example. Reparametrize by arc length:

$\vec{p}(t) = 3\sin(t)\vec{i} + 4t\vec{j} + 3\cos(t)\vec{k}$
for $t \geq 0$

Example. Reparametrize by arc length:

$\vec{r}(t) = e^{3t}\vec{i} + e^{3t}\vec{j} + 3\vec{k}$
for $t \geq 0$