§14.1 Vector Functions

After completing this section, students should be able to:

• Define a vector valued function.
• Find the domain of a vector valued function.
• Find the limit of a vector valued function.
• Match equations of vector valued functions with their graphs by considering the projections of the graphs onto the $xy$, $yz$, and $xz$ planes.
• Give a vector valued equation for the intersection of two surfaces.
Definition. A vector function or vector-valued function is:

If we think of the vectors as position vectors with their initial points at the origin, then the endpoints of \( \vec{v}(t) \) trace out a ______ in \( \mathbb{R}^3 \) (or in \( \mathbb{R}^2 \)).
Example. Sketch the curve defined by the vector function \( \mathbf{r}(t) = \langle t, \sin(5t), \cos(5t) \rangle \).
Example. Consider the vector function \( \mathbf{r}(t) = \frac{t^2 - t}{t - 1} \mathbf{i} + \sqrt{t + 8} \mathbf{j} + \frac{\sin(\pi t)}{\ln t} \mathbf{k} \)

1. What is the domain of \( \mathbf{r}(t) \)?

2. Find \( \lim_{t \to 1} \mathbf{r}(t) \)

3. Is \( \mathbf{r}(t) \) continuous on \((0, \infty)\)? Why or why not?
Review. Which of these are vector functions?

A. \( f(t) = t^2 \)
B. \( f(s, t) = 3x - 4t \)
C. \( f(t) = t^2 \vec{i} - 2t \vec{j} + \sqrt{t} \vec{k} \)
Review. Match the vector functions with the curves.

1. $\mathbf{r}_1(t) = < t^2, t^4, t^6 >$
2. $\mathbf{r}_2(t) = < t + 2, 3 - t, 2t - 1 >$
3. $\mathbf{r}_3(t) = < \cos(t), -\cos(t), \sin(t) >$
4. $\mathbf{r}_4(t) = < t, t^2, t^3 >$
5. $\mathbf{r}_5(t) = < \cos(t), \sin(t), t >$
6. $\mathbf{r}_6(t) = < \cos(t), \sin(t), \cos(2t) >$
Example. Consider the vector function $\mathbf{r}(t) = te^{-t}\mathbf{i} + \frac{t^3 + t}{2t^3 - 1}\mathbf{j} + \frac{1}{\sqrt{t}}\mathbf{k}$

1. What is the domain of $\mathbf{r}(t)$?

2. Find $\lim_{t \to \infty} \mathbf{r}(t)$
Example. Find the point on the curve \( \vec{r}(t) = 5 \cos(t) \hat{i} + 3 \sin(t) \hat{j} + 4 \sin(t) \hat{k} \) that lies closest to the point \( P(1, 1, 2) \).
Example. At what points does the helix \( \vec{r}(t) = < \sin(t), \cos(t), t > \) intersect the sphere \( x^2 + y^2 + z^2 = 5 \)?
Example. Show that the curve $\vec{r}(t) = 3 \cos(t)\hat{i} + 9 \cos(2t)\hat{j} + 3 \sin(t)\hat{k}$ lies on the intersection of the hyperboloid $y = x^2 - z^2$ and the cylinder $x^2 + z^2 = 9$. 
Extra Example. Find a function \( \vec{r}(t) \) that describes the curve where the following surfaces intersect.

\[ z = 3x^2 + y^2 + 1, \quad z = 5 - x^2 - 3y^2 \]
Extra Example. Find the curve where the following surfaces intersect.

\[ x^2 + y^2 = 25, \quad z = 2x + 2y \]
Extra Example. Find the curve where the following surfaces intersect.

\[ z = y + 1, \quad z = x^2 + 1 \]
Extra Example. §14.2 Derivatives and Integrals of Vector Functions

By the end of this section, students should be able to:

• Compute the derivative of a vector function.
• Compute the integral of a vector function.
• When \( \vec{r}(t) \) represents the position of a particle at time \( t \), explain the meaning of \( \vec{r}'(t) \), its direction, and its magnitude.
Suppose a particle is moving according to the vector equation \( \mathbf{r}(t) \). How can we find a tangent vector that gives the direction and speed that the particle is traveling?
**Definition.** The *derivative* of the vector function \( \mathbf{r}(t) \) is the same thing as the tangent vector, defined as

\[
\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) =
\]

If \( \mathbf{r}(t) = \langle r_1(t), r_2(t), r_3(t) \rangle \), then

\[
\mathbf{r}'(t) =
\]

The derivative of a vector function is a (circle one) vector / scalar.

The **unit tangent vector** is:

\( \mathbf{T}(t) = \)

The **tangent line** at \( t = a \) is:
Example. For the vector function $\vec{r}(t) = < t^2, t^3 >$

1. Find $\vec{r}''(1)$.
2. Sketch $\vec{r}(t)$ and $\vec{r}''(1)$.
3. Find $\vec{T}(1)$.
4. Find the equation for the tangent line at $t = 1$. 
Definition. If \( \vec{r}(t) = < f(t), g(t), h(t) > \), then

\[
\int \vec{r}(t) \, dt =
\]

and

\[
\int_a^b \vec{r}(t) \, dt =
\]
Example. Compute \( \int_1^2 \frac{1}{t} \mathbf{i} + e^t \mathbf{j} + te^t \mathbf{k}. \)
Review. If \( \vec{r}(t) = 5t^2 \hat{i} + \sin(t) \hat{j} - 3\hat{k} \), how do we compute \( \vec{r}'(t) \)?

Question. For a vector function \( \vec{r}(t) = < r_1(t), r_2(t), r_3(t) > \), what is the limit definition of \( \vec{r}''(t) \)?
Question. (Geometric interpretation) If we think of $\mathbf{r}(t)$ as a space curve, what does $\mathbf{r}''(t)$ represent geometrically?

Question. (Physics interpretation) If $\mathbf{r}(t)$ represents the position of a particle at time $t$, (a) what does the direction of $\mathbf{r}''(t)$ signify?

(b) what does the magnitude of $\mathbf{r}''(t)$ signify?

Note. The *unit tangent vector* is computed as ____________ and sometimes denoted by ____________ .
Example. Find the tangent vector, the unit tangent vector, and the tangent line for the following curves at the point given

1. \( \vec{r}(t) = \langle t, t^2, t^3 \rangle \) at \( t = 1 \)

2. \( \vec{r}(t) = \langle t^2, t^4, t^6 \rangle \) at \( t = 1 \)

3. \( \vec{p}(t) = \langle t + 2, 3 - t, 2t - 1 \rangle \) at \( t = 0 \)
Example. At what point do the curves $\mathbf{r}_1(t) = \langle t, 1-t, 3+t^2 \rangle$ and $\mathbf{r}_2(t) = \langle 3-t, t-2, t^2 \rangle$ intersect? Find their angle of intersection correct to the nearest degree.
Derivative rules - see textbook

- Is there a product rule for derivatives of vector functions?
- Is there a quotient rule for derivatives of vector functions?
- Is there a chain rule for derivatives of vector functions?
Example. Show that if $||\vec{r}(t)|| = c$ (a constant), then $\vec{r}''(t)$ is orthogonal to $\vec{r}(t)$ for all $t$. 
Review. If \( \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \), then

\[
\int \mathbf{r}(t) \, dt =
\]

and

\[
\int_a^b \mathbf{r}(t) \, dt =
\]

Example. Find \( \mathbf{p}(t) \) if \( \frac{d}{dt}(\mathbf{p}) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + tk \) and \( \mathbf{p}(1) = 6\mathbf{i} + 6\mathbf{j} + 6\mathbf{k} \).
Extra Example. Show that if \( \vec{r} \) is a vector function such that \( \vec{r}'' \) exists, then

\[
\frac{d}{dt} [\vec{r}'(t) \times \vec{r}''(t)] = \vec{r}'(t) \times \vec{r}'''(t)
\]

Extra Example. If \( \vec{u}(t) = \vec{r}(t) \circ [\vec{r}'(t) \times \vec{r}''(t)] \), show that

\[
\vec{u}''(t) = \vec{r}(t) \cdot [\vec{r}'(t) \times \vec{r}'''(t)]
\]
§14.4 Arclength

After completing this section, students should be able to:

- Set up an integral to represent the arclength of a curve, and compute the integral when it simplifies nicely.
- Explain what it means for a curve to be parametrized by arclength.
- Reparametrize curves so that they are parametrized by arclength.
Example. Find the length of this curve.
**Note.** In general, it is possible to approximate the length of a curve $x = f(t)$, $y = g(t)$ between $t = a$ and $t = b$ by dividing it up into $n$ small pieces and approximating each curved piece with a line segment.

Arc length is given by the formula:
Set up an integral to express the arclength of the Lissajous figure

\[ x = \cos(t), \quad y = \sin(2t) \]
Review. To find the arc length of a space curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, we can approximate it with straight line segments.

Note. The arc length of a curve $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ between $t = a$ and $t = b$ is given by
Definition. The arc length function (starting at $t = a$) is

$$s(t) =$$

Note. If $s(t)$ is the arc length function, then $s'(t) =$

In words, this says that the rate of change of the arclength with respect to time is ...
Example. Consider the two curves:

1. \( \vec{r}(u) = \langle 2u, u^2, \frac{1}{3}u^3 \rangle \) for \( 0 \leq u \leq 1 \)

2. \( \vec{q}(t) = \langle 2 \ln(t), (\ln(t))^2, \frac{1}{3}(\ln(t))^3 \rangle \) for \( 1 \leq t \leq e \)

How are the curves related?

We say that \( \vec{q}(t) \) is a reparametrization of \( \vec{r}(u) \) because:

Also \( \vec{r}(u) \) is a reparametrization of \( \vec{q}(t) \) because:

You can think of a reparametrization of a curve as the same curve, traveled at a different speed. In our case, \( \vec{q} \) moves along the curve (circle one) slower / faster than \( \vec{r} \).

In mathematical notation, \( \vec{q}(t) \) is a reparametrization of \( \vec{r}(u) \) if \( \vec{q}(t) = \vec{r}(\phi(t)) \) for some strictly increasing (and therefore invertible) function \( u = \phi(t) \).
Find the arc length of each curve.

\[ \vec{r}(u) = <2u, u^2, \frac{1}{3}u^3> \]
for \(0 \leq u \leq 1\)

\[ \vec{q}(t) = <2\ln(t), (\ln(t))^2, \frac{1}{3}(\ln(t))^3> \]
for \(1 \leq t \leq e\)
Fact. Arc length does not depend on parametrization.

Proof:

Is there a natural, best way to parametrize a curve?
**Definition.** We say that a curve $\vec{r}(t)$ is parametrized by arclength if ....

**Note.** If the curve $\vec{r}(t)$ is parametrized by arclength then ...

**Note.** If $||\vec{r}'(t)|| = 1$ for all $t$, then ...
Example. Reparametrize by arc length:

\[ \vec{p}(t) = 3 \sin(t) \vec{i} + 4t \vec{j} + 3 \cos(t) \vec{k} \]

for \( t \geq 0 \)
Example. Reparametrize by arc length:

\[ \vec{r}(t) = e^{3t} \vec{i} + e^{3t} \vec{j} + 3\vec{k} \]

for \( t \geq 0 \)