§10 Finding All Types of Isometries (Optional)

The goal of this section is to prove that there are only four (or five) isometries of the plane:

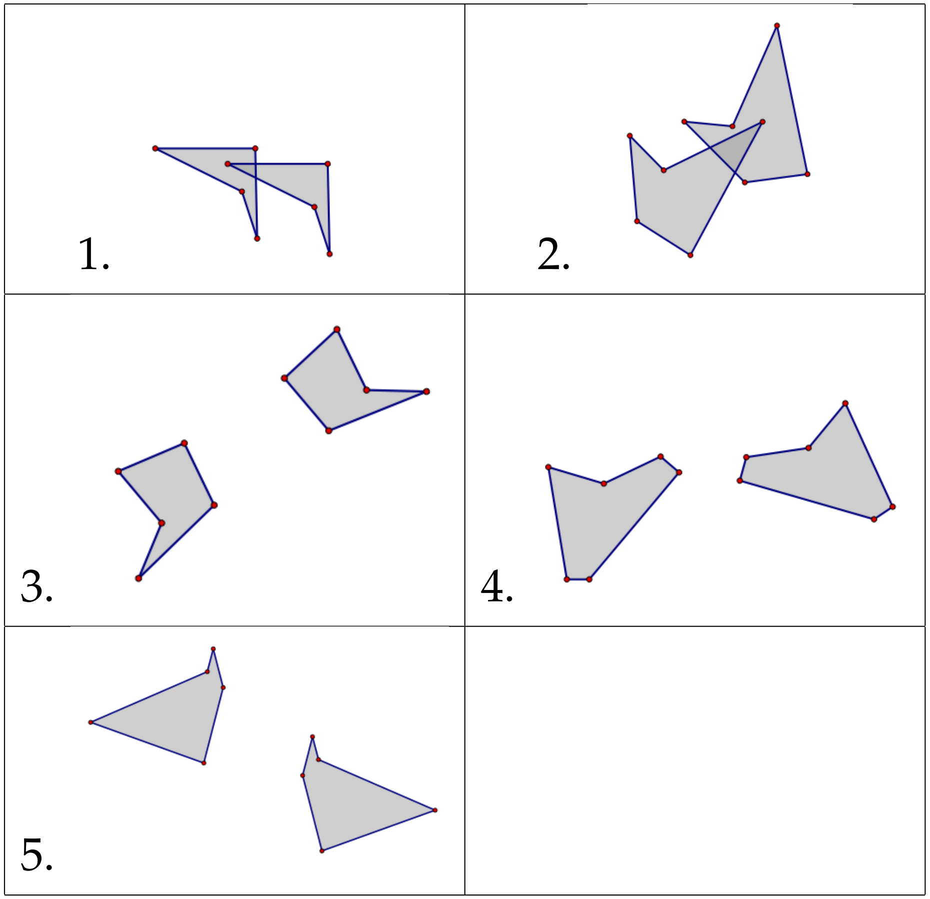
Reference: *Groups and Symmetry: A Guide to Discovering Mathematics* by David Farmer

Supplies:

- Geogebra (phones or laptops)

Factoring an isometry as a product of refelctions

Each figure below shows a pair of isometric (congruent) polygons.



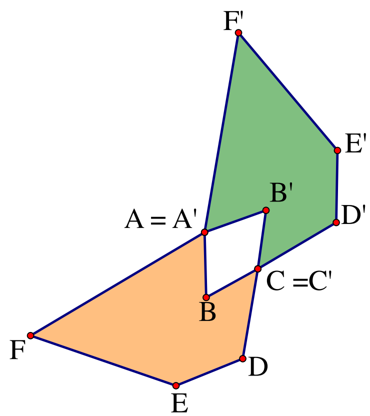
For each figure, find a sequence of no more than *three* reflections so that one polygon ends up coinciding with the other polygon.

- Can *every* isometry can be viewed as a product of either 1, 2, or 3 reflections?
  - If so, find an *algorithm* to locate the mirrors (i.e. give instructions that could be applied to any pair of isometric figures).
  - If not, draw a pair of isometric figures for which there is no such sequence.

- For the dinosaur stamp symmetry page, write the isometry that gets from dinosaur A to dinosaur B as a product of at most 3 reflections, by using Geogebra to draw the exact mirror lines. Do the same thing for the isometry that gets from A to C and for the isometry that gets from B to C.

Polygons that share three vertices

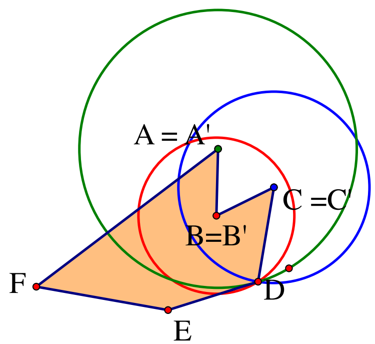
A *vertex* of a polygon is a corner where two edges meet.



These two polygons share two vertices.

- Is it possible to draw two congruent polygons in the plane that share three corresponding vertices but don't share all their vertices?
- What about for polyhedra in 3-dimensional space?

Theorem. If two congruent polygons share three corresponding vertices that are not colinear, then they share all vertices.



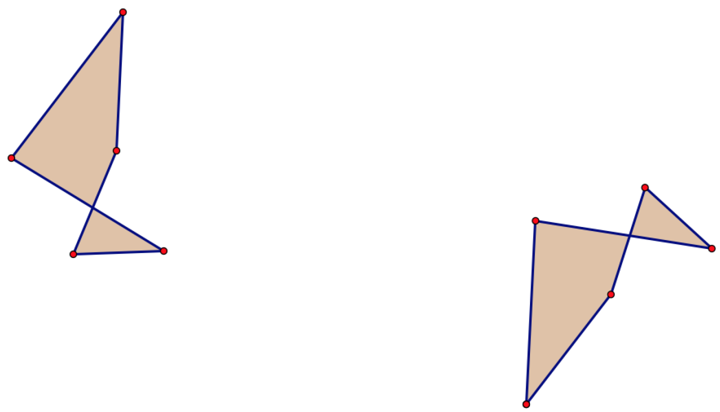
Proof. • Step 1: Suppose we have two congruent polygons that share three corresponding vertices A, B, and C.

• Step 2: Suppose there is a fourth vertex D on the first polygon.

• To continue this proof, a hint: draw circles.

□

Given any isometry, how can we write it as a product of three reflections ... ... EVEN IF we don't know ahead of time that it is a translation, reflection, rotation, or glide?



HINT: Perform one reflection at a time to "fix up" one point at a time. That is, first reflect to bring one point A onto its image A', then reflect to bring another point B onto its image B', etc.

How many points to you need to "fix up" before you are done?

Theorem. Any isometry can be written as a product of at most three reflections.

Proof.
- Pick three non-collinear points A, B, and C and find the points A', B', C', where the isometry takes them.

- It is also possible to find a product of three reflections that takes A to A', B to B', and C to C'. How?

- Continue the proof from here.

□

What about in 3-d?

Theorem. There are no other isometries of the plane besides ...

Proof.
- Any isometry of the plane can be written as the product of one, two, or three reflections.
- If only one reflection is needed, then the isometry is a reflection.
- If exactly two reflections are needed, then the isometry is:
- If exactly three reflections are needed, then the isometry is:
- Why is the last statement true?

□

Why is a product of three reflections always a reflection or a glide reflection?

- Suppose we have an isometry that is a product of three reflections through mirrors $m1$, $m2$, and $m3$.
- If $m1$, $m2$, and $m3$ are all parallel, then the product of the reflections is a ____________ . Why?
- If $m2$ and $m3$ intersect, then reflection through $m2$ and then $m3$ is a ________ with rotocenter at the intersection of $m2$ and $m3$.
- So we can think of our isometry as reflection through $m1$ followed by rotation around this intersection point.
- But if we rotate $m2$ and $m3$ around this intersection point, we'll still get the same rotation with the same rotocenter.
- So rotate $m2$ and $m3$ around their intersection point until $m2$ is perpendicular to $m1$.
- Now our isometry is reflection through $m1$, then $m2$, then $m3$, and $m1$ is perp to $m2$.
- So reflection through $m1$ then $m2$ is the same as rotation by ________ degees with rotocenter at the intersection of $m1$ and $m2$.
- If we rotate $m1$ and $m2$ around their intersection point, we still get the same rotation.

- So rotate $m1$ and $m2$ around their intersection point until $m2$ is parallel to $m3$.
- Now we have $m2$ and $m3$ parallel, and $m1$ perpendicular to both.
- Why is this a glide reflection?

There are a few small details to worry about

- If $m1$ and $m2$ and $m3$ all intersect in the same point, then we need to modify the argument. How?
- If $m2$ and $m3$ are parallel, we need to modify the argument. How?