§10 Finding All Types of Isometries (Optional)

The goal of this section is to prove that there are only four (or five) isometries of the plane:

Reference: *Groups and Symmetry: A Guide to Discovering Mathematics* by David Farmer

Supplies:
- Geogebra (phones or laptops)
Factoring an isometry as a product of reflections

Each figure below shows a pair of isometric (congruent) polygons.

For each figure, find a sequence of no more than three reflections so that one polygon ends up coinciding with the other polygon.
• Can every isometry can be viewed as a product of either 1, 2, or 3 reflections?
  – If so, find an algorithm to locate the mirrors (i.e. give instructions that could be applied to any pair of isometric figures).
  – If not, draw a pair of isometric figures for which there is no such sequence.
For the dinosaur stamp symmetry page, write the isometry that gets from dinosaur A to dinosaur B as a product of at most 3 reflections, by using Geogebra to draw the exact mirror lines. Do the same thing for the isometry that gets from A to C and for the isometry that gets from B to C.
Polygons that share three vertices

A vertex of a polygon is a corner where two edges meet.

These two polygons share two vertices.

- Is it possible to draw two congruent polygons in the plane that share three corresponding vertices but don’t share all their vertices?
- What about for polyhedra in 3-dimensional space?
**Theorem.** If two congruent polygons share three corresponding vertices that are not colinear, then they share all vertices.

**Proof.**

- Step 1: Suppose we have two congruent polygons that share three corresponding vertices $A$, $B$, and $C$.
- Step 2: Suppose there is a fourth vertex $D$ on the first polygon.
- To continue this proof, a hint: draw circles.
Given any isometry, how can we write it as a product of three reflections ... ... EVEN IF we don’t know ahead of time that it is a translation, reflection, rotation, or glide?

HINT: Perform one reflection at a time to "fix up" one point at a time. That is, first reflect to bring one point \( A \) onto its image \( A' \), then reflect to bring another point \( B \) onto its image \( B' \), etc.

How many points do you need to "fix up" before you are done?
Theorem. Any isometry can be written as a product of at most three reflections.

Proof. • Pick three non-collinear points $A$, $B$, and $C$ and find the points $A'$, $B'$, $C'$, where the isometry takes them.

• It is also possible to find a product of three reflections that takes $A$ to $A'$, $B$ to $B'$, and $C$ to $C'$. How?

• Continue the proof from here.

What about in 3-d?
Theorem. There are no other isometries of the plane besides ...

Proof. • Any isometry of the plane can be written as the product of one, two, or three reflections.
  • If only one reflection is needed, then the isometry is a reflection.
  • If exactly two reflections are needed, then the isometry is:
  • If exactly three reflections are needed, then the isometry is:
    • Why is the last statement true?
Why is a product of three reflections always a reflection or a glide reflection?

- Suppose we have an isometry that is a product of three reflections through mirrors $m_1$, $m_2$, and $m_3$.
- If $m_1$, $m_2$, and $m_3$ are all parallel, then the product of the reflections is a ______. Why?
- If $m_2$ and $m_3$ intersect, then reflection through $m_2$ and then $m_3$ is a ______ with rotocenter at the intersection of $m_2$ and $m_3$.
- So we can think of our isometry as reflection through $m_1$ followed by rotation around this intersection point.
- But if we rotate $m_2$ and $m_3$ around this intersection point, we’ll still get the same rotation with the same rotocenter.
- So rotate $m_2$ and $m_3$ around their intersection point until $m_2$ is perpendicular to $m_1$.
- Now our isometry is reflection through $m_1$, then $m_2$, then $m_3$, and $m_1$ is perp to $m_2$.
- So reflection through $m_1$ then $m_2$ is the same as rotation by ______ degrees with rotocenter at the intersection of $m_1$ and $m_2$.
- If we rotate $m_1$ and $m_2$ around their intersection point, we still get the same rotation.
• So rotate \( m_1 \) and \( m_2 \) around their intersection point until \( m_2 \) is parallel to \( m_3 \).

• Now we have \( m_2 \) and \( m_3 \) parallel, and \( m_1 \) perpendicular to both.

• Why is this a glide reflection?
There are a few small details to worry about

- If $m_1$ and $m_2$ and $m_3$ all intersect in the same point, then we need to modify the argument. How?
- If $m_2$ and $m_3$ are parallel, we need to modify the argument. How?