§23 Hyperbolic Geometry

The goal for this part is to become more familiar with negatively curved surfaces and hyperbolic geometry.

References:

- Ideas, templates, and instructions are from Frank Sottile `http://www.math.tamu.edu/~sottile/research/stories/hyperbolic_football/`

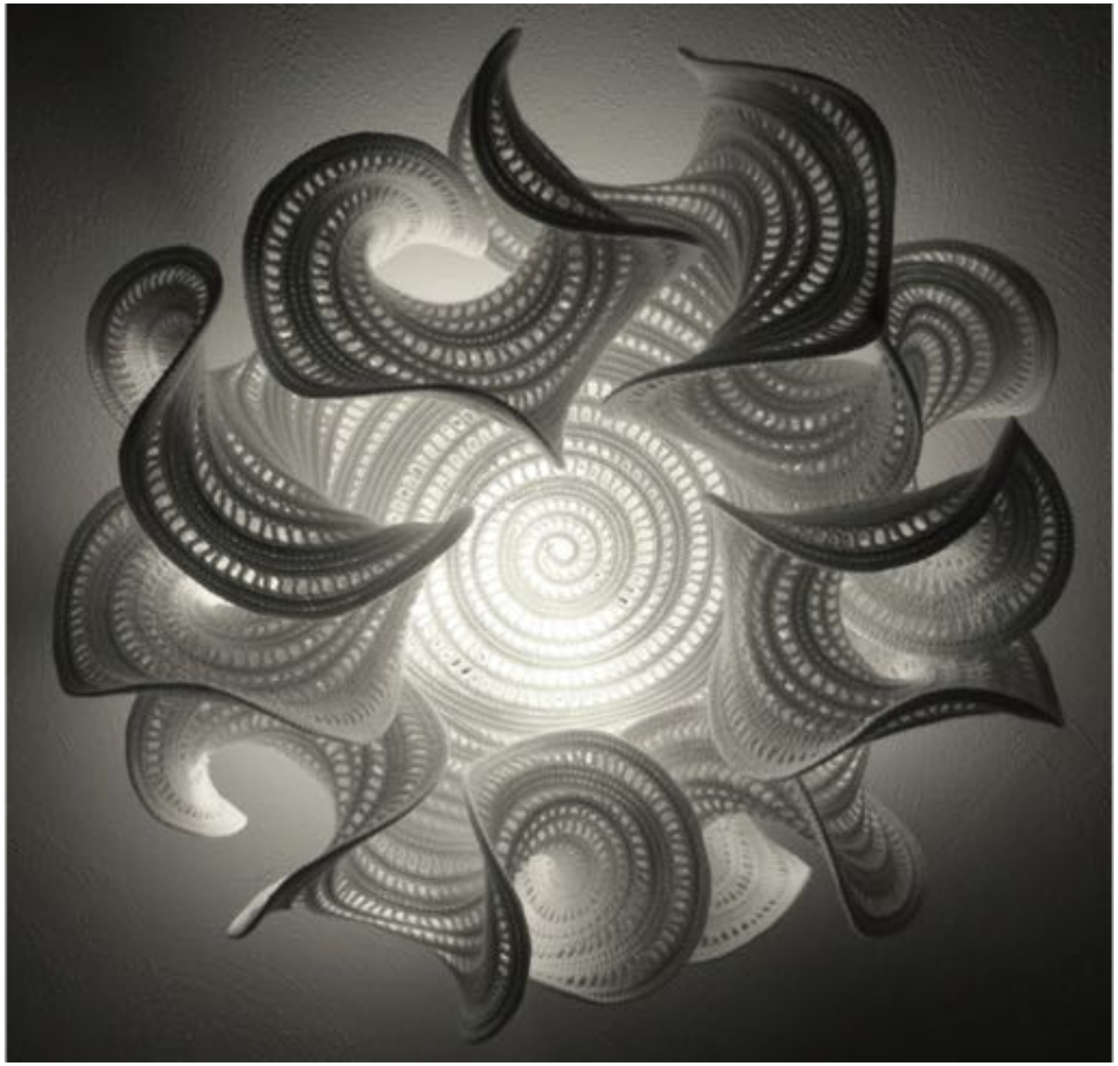
Supplies:

- Templates
- Scissors
- Tape
- index cards
- small or flexible rulers

Negatively curved surfaces in nature

Negatively curved surfaces in art

by Daina Taimina



by Gabriele Meyer

by Gabriele Meyer

Hyperbolic soccerballs

Recall that a standard soccerball has one black pentagon and two white hexagons at each vertex.

The angle defect at each vertex is:

What is the angle defect if we use one black heptagon (7-sided polygon) and two white hexagons at each vertex?

Use the templates to tape together one black heptagon and two white hexagons at each vertex.

Hints:

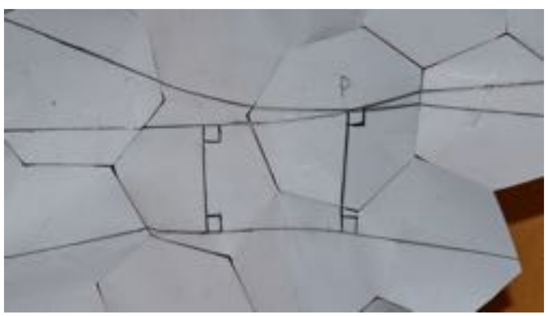
- Do as little cutting as possible. Don't separate all the polygons. Instead cut around the outside of a patch of hexagons, and separate only when necessary to insert a heptagon or more hexagons.
- Tape together along edges only. Don't try taping over vertices, where there is curvature!
- It's easier to work with a partner.

Hyperbolic geometry

In (flat) Euclidean geometry, given any straight line, and any point not on the line, there is exactly one parallel line through that point.

What is the nature of parallel lines on a negatively curved surface?

1. To draw a line on the model, flatten part of the model to start the line, and use a short (< 15 cm) straightedge to continue the line across the model, flattening pairs of polygons as needed. Try to avoid running the line through a vertex.
2. After drawing your line completely across the model, you can pick it up, straighten it along the line, and sight down the line to see that it is straight.
3. After drawing a first line, pick a point on it and draw a short line segment m perpendicular to it. Then start a new line perpendicular to m and extend this third line across the model. What do you notice about your original line and this new, parallel line?
4. On one of the parallel lines from the previous step, choose a point P not lying on the common perpendicular m. Dropping a perpendicular from P to the other line, and then taking a perpendicular to that through P gives a second line through P that is parallel to the original line.



1. Now, try to draw a triangle. For this it is best to try to make a big triangle.
2. Measure its interior angles.
   - One method is to mark off an arc on a small sector of a circle (cut out of a scrap of paper), lay the semicircle on a flat surface, and then use your protractor
   - It is possible to find the sum of the three angles by marking off three consecutive arcs along a small sector of a circle.
3. What is the sum of the interior angles $\alpha + \beta + \gamma$ of the triangle?
4. How far does the sum differ from 180°?
5. What is the total angle defect from the three vertices α, β, and γ of the triangle?

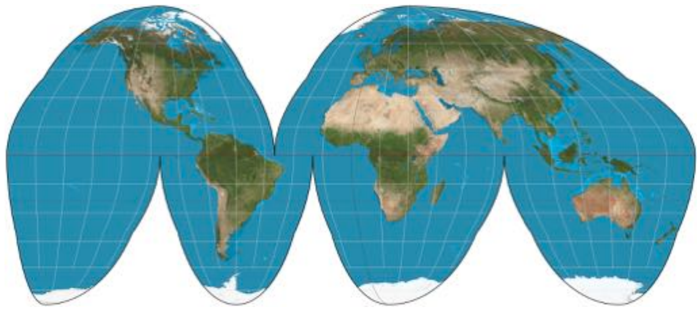
- Gauss-Bonnet says that the angle defect from the three vertices of the triangle, plus the curvature from the interior of the triangle, should add up to what number?
- Use Gauss-Bonnet to show that $180° - (\alpha + \beta + \gamma)$ should be equal in magnitude to the curvature in the interior of the triangle.
- Verify that this is true by counting up the curvature from the interior of the triangle.
- Draw a triangle that encloses as much curvature as possible.
- Is there a maximum amount of curvature that a hyperbolic triangle can contain?

Maps and projections

- It is hard to work with curly, floppy hyperbolic surfaces.
- So it can be helpful to represent the ”hyperbolic plane” on flat paper.
- With positively curved surfaces (like orange peels), it is not possible to flatten the surface without stretching it or tearing it.
- With negatively curved surfaces, it is not possible to flatten the surface without ...

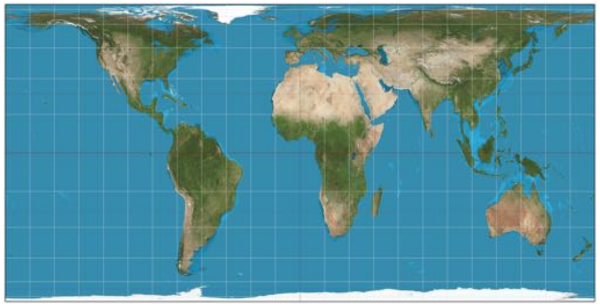
Projections of the sphere

To represent the spherical earth on a flat sheet of paper, you have to make some compromises:

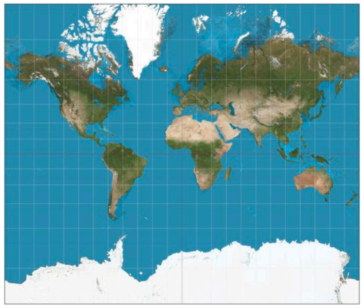


Goode-Homolsine Projection

Gnomonic Projection



Hobo-Dyer Projection



Mercator Projection

See also Wikipedia list of map projections

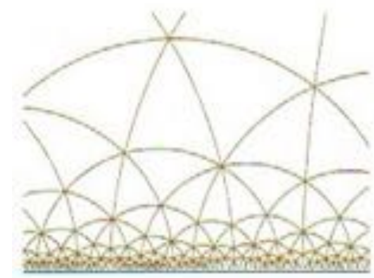
Maps of the hyperbolic plane

Here are three "projections" of the hyperbolic plane.

The Klein Model The Poincare Disk Model Upper Half Plane Model

Note all the yellow flowers are actually the same size and shape.

These are three representations of the same pattern of straight lines on the hyperbolic plane.



The Klein Model   The Poincare Disk Model   Upper Half Space Model

What do you think are some of the advantages and disadvantages of each projection?

Experiment with the Poincare Disk on this geogebra file: https://www.geogebra.org/m/R5e9AggU

- Draw a hyperbolic line segment and find its midpoint. Which point does the midpoint look like it is closer to, in your Euclidean perspective? How could you have predicted this?
- Construct a line and two lines parallel to it that both go through the same point. Hint: you did this before on the hyperbolic soccerball.
- Construct a hyperbolic triangle and calculate the sum of its angles. Be sure to use the hyperbolic angle tool! Try to draw a triangle for which the angle sum deviates from 180° as much as possible!

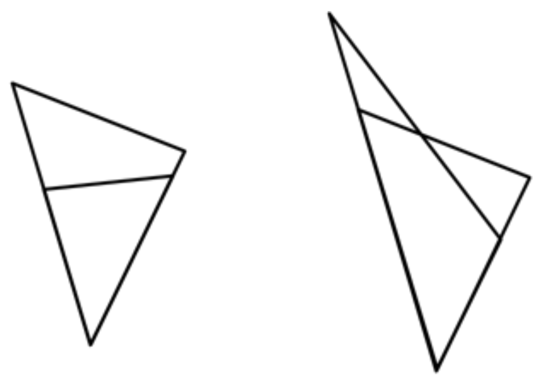
Download Hyperbolic Games at http://www.geometrygames.org/HyperbolicGames/ and play a few games.

Practice Problems

The hyperbolic soccerball is an approximation to a true hyperbolic plane, much in the same way that a flat-sided icosohedron (or a flat-sided soccerball polyhedron) is an approximation of a sphere.

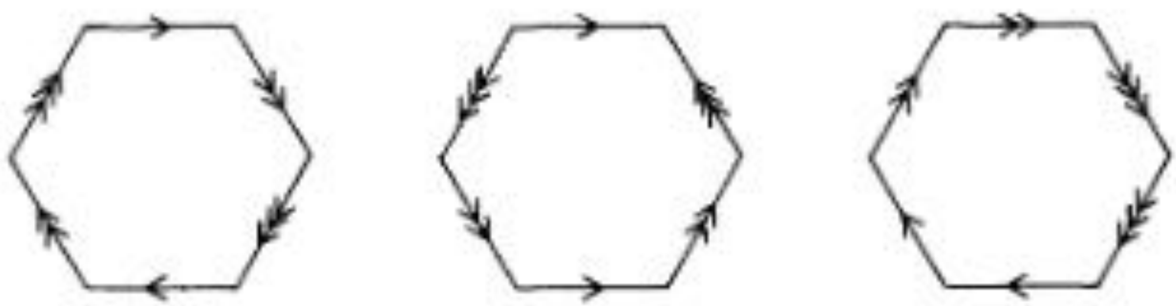
1. What does the Gauss-Bonnet Theorem tell you about the relationship between the area of a triangle drawn on a sphere of radius 1 compared to the angles α, β, and γ in the triangle? What is the sphere has radius r instead of 1?
2. What does the Gauss-Bonnet Theorem tell you about the relationship between the area of a triangle drawn on a true hyperbolic plane of constant curvature 1 compared to the angles α, β, γ in the triangle? What if the hyperbolic plane has constant curvature of κ instead of 1?
3. What is the area of a triangle with all three angles equal to $90°$ drawn on the sphere of radius 1?
4. What is the area of a triangle with angles $30°$, $45°$, $90°$ on a hyperbolic plane of constant curvature 1?
5. Do the angles of a triangle always add up to the same number on the hyperbolic plane? Is there a maximum angle sum? A minimum angle sum? What about on the sphere?

6. Is there a maximum area for a triangle you can draw on the sphere? If so, what?
7. Is there a maximum area for a triangle you can draw on the hyperbolic plane?
8. What is the maximum angle sum for a quadrilateral on the hyperbolic plane?
9. In hyperbolic geometry, prove that two triangles must be congruent if the measures of their corresponding angles are equal. Is the same statement true for spherical geometry and Euclidean geometry? Hint: If the triangles are not congruent, then it is possible to line up the triangles so that one is inside the other, like in the left figure, rather than one overlapping the other as in the right figure. Why? Now think in terms of angles or area.



Constant curvature geometries

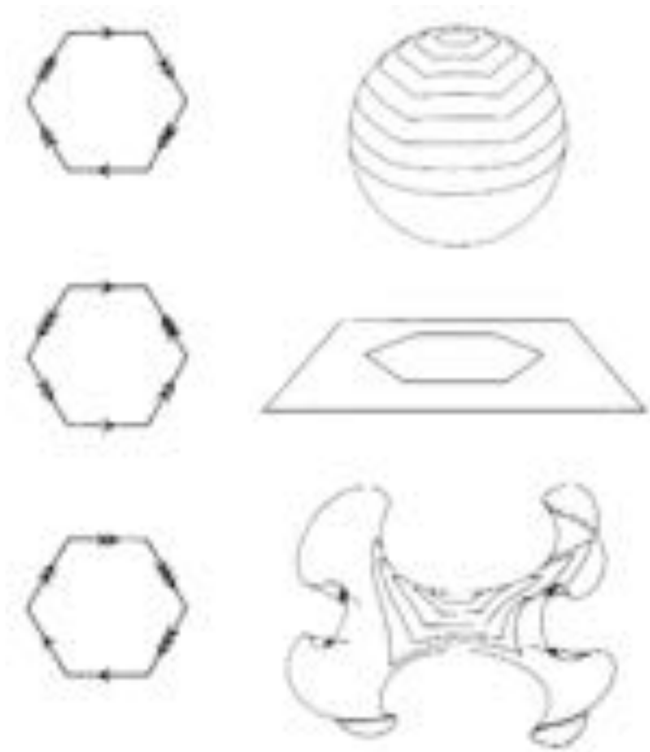
The figure below shows three surfaces, each a haxagon with edges glued as indicated by the arrows. Use the "walking around the corners" technique to decided how each hexagon's corners fit together. Which of these suraces have cone points (points of angle defect)? Which have points of angle excess?



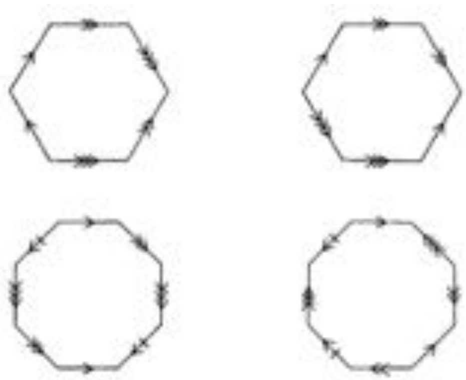
One of these three surfaces has no cone points and no points of angle excess. This surface has 0 curvature everywhere – it has flat, Euclidean geometry. Which of the three surfaces is it? What is the common name for this surface?

One of these three surfaces has cone points but no points of angle excess. If we place the hexagon on the top of a sphere and make it grow larger and larger, it will get larger and larger angles. So to get rid of the cone points, we need to make the hexagon big enough that the cone points disappear. This happens when each angle is 180° and the hexagon fills an entire hemisphere. We have now given the surface homogeneous spherical geometry. Which of the three surfaces is this and what is its common name?

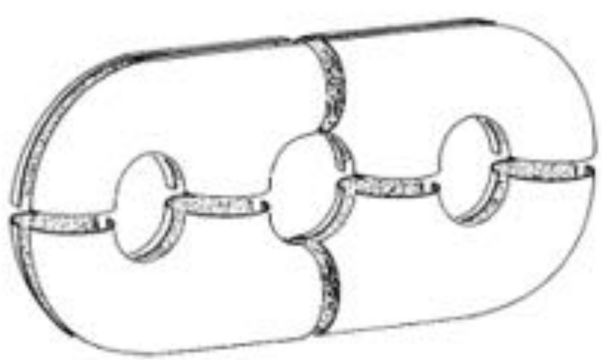
To get rid of points of angle excess, we have to put the hexagon on a hyperbolic plane, where larger polygons have smaller angles. If we let the haxagon grow until each angle shrinks to 60°, then the points of angle excess with disappear, and the surface will have homogeneous hyperbolic geometry. Which of the three surfaces is this and what is its common name?



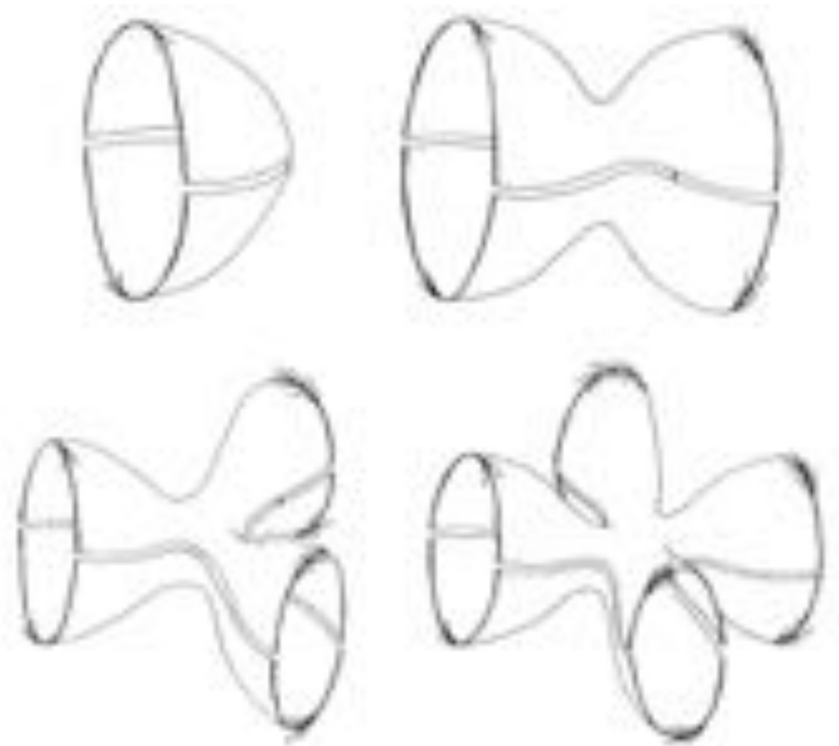
Exercise. For each surface in the figure below, determine how the polygon's corners fit together and whether the surface has cone points. Which of the surfaces can be given spherical geometry, which can be given Euclidean geometry, and which can be given hyperbolic geometry?



Exercise. Prove that a connected sum of n tori, for $n > 1$, can be given homogeneous hyperbolic geometry. Hint: cut it up into hexagons. How many hexagons do you get when you cut up an n-holed torus?



Exercise. The connected sum of two projective planes can be cut into two squares. Similarly, $P\#P\#P$ can be cut into two hexagons, $P\#P\#P\#P$ can be cut into two octagons, etc. In each case, the polygon's corners meet in groups of four (two corners on one side of a rim are glued to the two corners on the opposite side). Which of these surfaces can be given which homogeneous geometry?



Exercise. Recall that every surface (without boundary) is topologically equivalent to a surface on the following list:

$$\begin{array}{llllll} & T^2 & T^2 \# T^2 & T^2 \# T^2 \# T^2 & T^2 \# T^2 \# T^2 \# T^2 & \ldots \\ S^2 & & & & & \\ & P^2 & P^2 \# P^2 & P^2 \# P^2 \# P^2 & P^2 \# P^2 \# P^2 \# P^2 & \ldots \end{array}$$

Place each surface on the list into an appropriate box in the table.

	orientable	nonorientable
elliptic		
Euclidean		
hyperbolic		