§25 Maps and Graphs

References:

- *Beginning Topology* by Sue Goodman, Chapter 4

Supplies:

- markers, colored pencils, or crayons

Map coloring

Definition. A *map* on a surface M is ...

Definition. To *color* a map is to ...

Definition. A map is *N-colorable* if ...

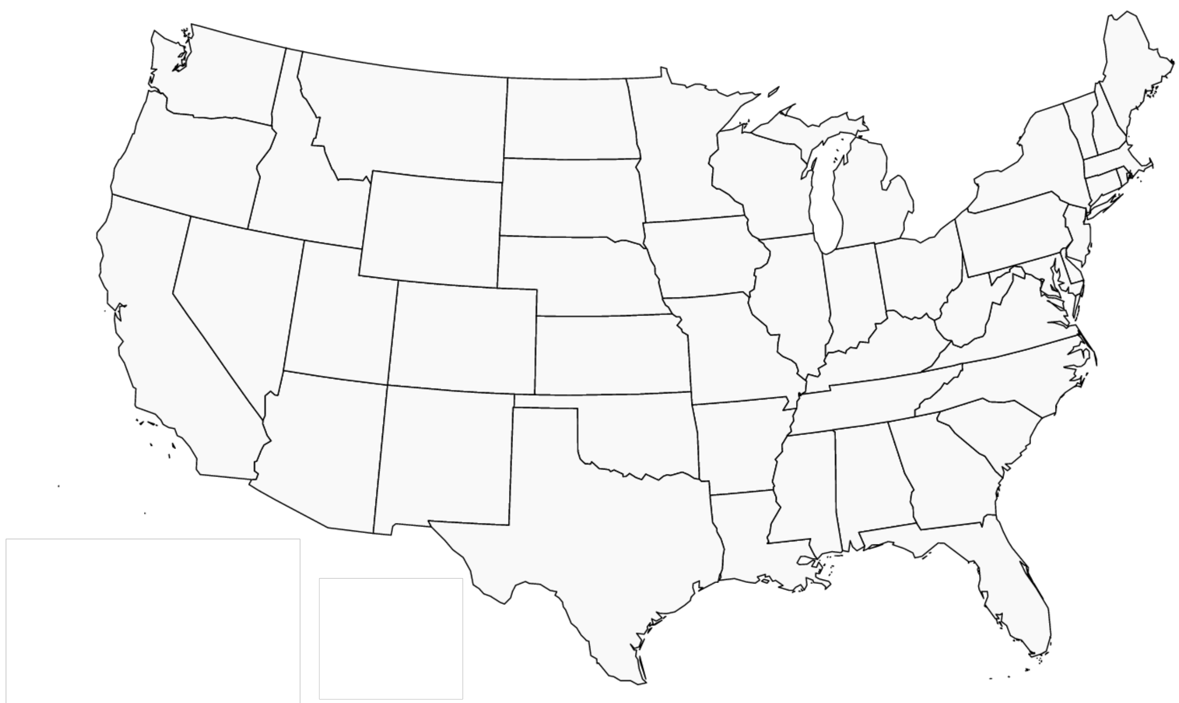
Definition. The *coloring number* of a surface M is ...

Note. We will primarily work with compact, connected surfaces without boundary.

Question. (Goal) What is the coloring number of the sphere? other surfaces?

Note. The coloring number of the plane is the same as the coloring number of the ...

Example. How many colors are needed to color this map?



Fact. It turns out that ______ colors are enough to color any map on the sphere. We will prove the weaker statement that ______ colors are enough.

Question. Is the coloring number for the torus the same as the coloring number for the sphere?

Question. How can we prove that a certain number of colors N will suffice?

Main Idea #1 Use induction on number of faces.

Proposition. Given a surface M, suppose that there is a positive integer N such that for any map on M, there is a face with ...
Then N colors suffice for all maps on M.

Corollary. Given a surface M, suppose that there is a positive integer N such that for any map on M the average number of edges per face ...
Then N colors suffice for all maps on M.

Corollary. Given a surface M, suppose that there is a positive integer N such that $\quad < N$. Then N colors suffice for all maps on M.

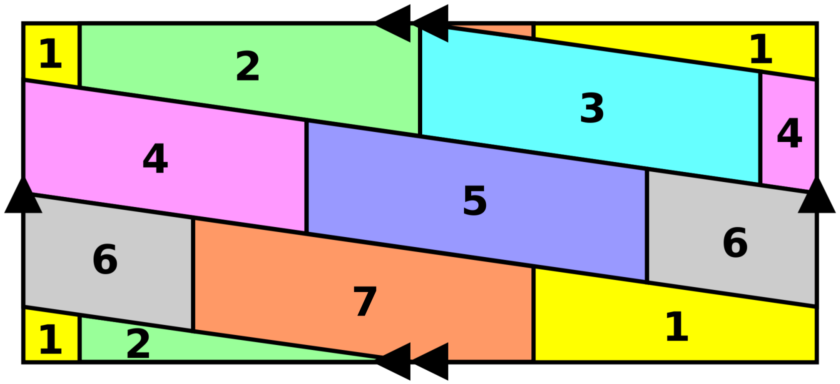
Main Idea #2 Use Euler characteristic.

Proposition. For any map on surface M, $\frac{2e}{f} \leq 6\left[1 - \frac{\chi(M)}{f}\right]$.

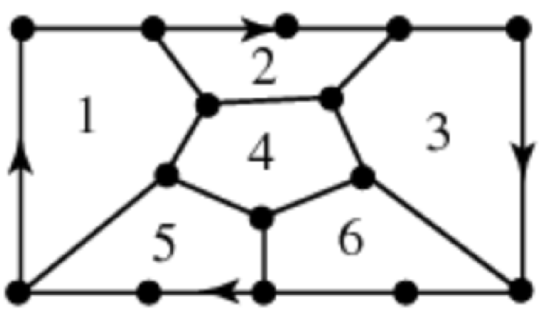
Theorem. Any map on S^2 or P can be colored with ________ colors.

Theorem. Any map on T or K can be colored with ________ colors.

In fact, there are maps on the torus that require 7 colors, so the coloring number of the torus is 7.



There are maps on the projective plane that require 6 colors, so the coloring number of the projective plane is 6.



There are no maps on the Klein bottle that require more than 6 colors, but we will not prove this.

Question. What if $\chi(M) < 0$? For what values of N can we guarantee that any map on M is N-colorable?

Theorem. (Haewood 1890) For a compact surface M with $\chi(M) \leq 0$, any map on M is N colorable where

$N =$

$\chi(M)$	-1	-2	-3	-4	-5	-6	-7
H_M							

$\chi(M)$	2	1	0
H_M			

Five Color Theorem for S^2

Goal: Prove that 5 colors are enough to color any map on S^2.

Note. It is enough to show that any *trivalent* map on S^2 can be 5-colored.

Lemma. Any trivalent map on the sphere has a face with fewer than six sides.

Proof. Homework

Theorem. Any trivalent map on the sphere (and therefore any map on the sphere) can be colored with at most 5 colors.

Use induction argument as before: base case, if $f \leq 5$ can color each face a different color. Assume that any trivalent map with k faces can be colored with 5 colors, and prove true for a trivalent map with $k + 1$ faces.

Case 1: Suppose there is a face A with 1, 2, 3, or 4 edges.

Case 2: Suppose there is a face A with 5 edges.

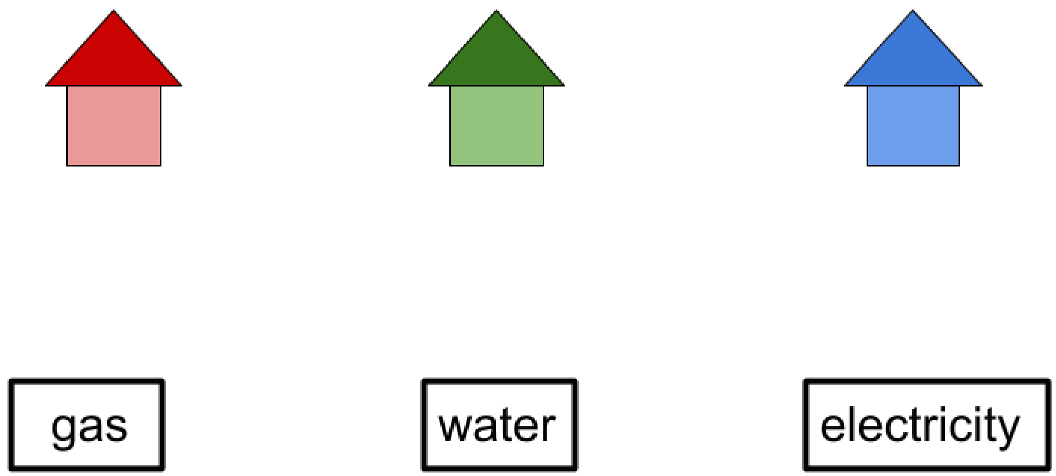
Subcase 2a: Suppose A's 5 neighbors are not actually all distinct faces.

Subcase 2b: Suppose A's 5 neighbors are all distinct faces.

Claim: There is a pair of neighbors of A that DO NOT share an edge.

Graphs in Surfaces

Example. Suppose there are three cottages on a plane and each needs to be connected to the gas, water, and electricity companies. Using a third dimension or sending any of the connections through another company or cottage is disallowed. Is there a way to make all nine connections without any of the lines crossing each other?

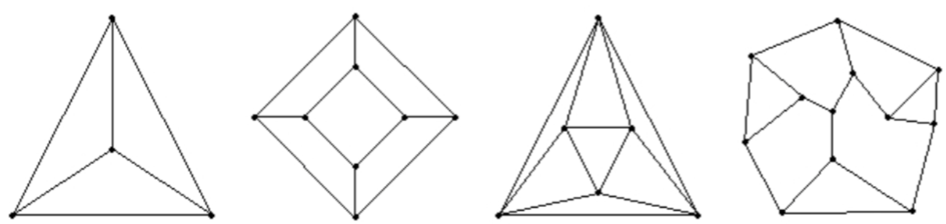


Definition. A graph is *embedded in a surface* if it can be realized as a subset of the surface so that ...

Unless otherwise specified, we will work with graphs that have no edge loops or parallel edges. We will also assume that our graphs are connected.

Definition. A graph is a *planar graph* if ...

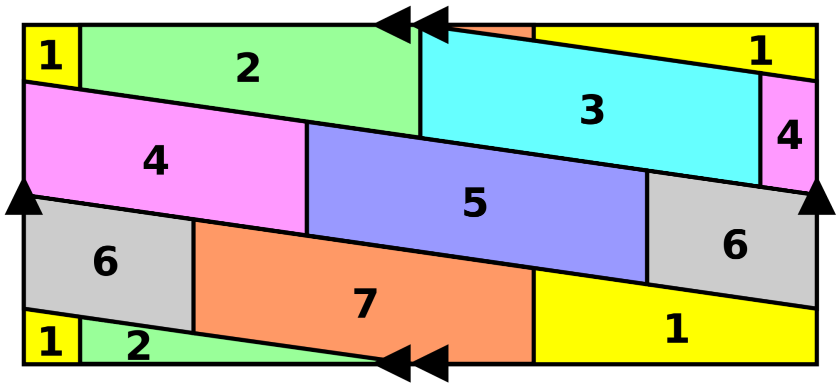
Equivalently, a graph is a *planar graph* if

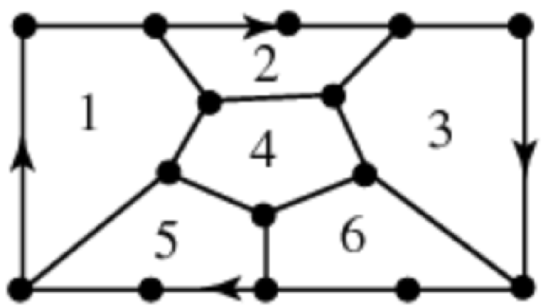


Goals: In this section, we will explore questions about what graphs can be embedded in what surfaces.

Note. Embedding graphs in surfaces is related to coloring maps on surfaces.

Question. Given a map on a surface, how can we produce an embedded graph on the surface?





Question. Given an embedded graph in a surface, how can we produce a map on a surface?

Note. Previously, we proved that for a surface M, the minimum number of colors needed to color *any* map on M is less than or equal to Haewood's number:

Actually we proved this for most surfaces, but only stated it and did not prove it for ...

In order to prove that the minimum number of colors needed is exactly Haewood's number (with one exception: the Klein bottle), we need to ....

One way to do this is to show that for any surface M (except the Klein bottle), the complete graph K_H can be embedded in M, where H is ...

This is actually true for all surfaces EXCEPT ...

We will actually prove that K_H can be embedded in M for ...

Note. Here is what we will do instead of finishing the proof that coloring number is given by Haewood's formula:

- Use Euler characteristic to prove that K_5 cannot be embedded in S^2.
- Use Euler characteristic to prove that $K_{3,3}$ cannot be embedded in S^2.
- State some facts about what graphs can and cannot be embedded in S^2.
- Use Euler characteristic to find a bound on the Euler characteristic of a surface that K_n embeds in.

Proposition. The complete graph K_5 does not embed in the sphere.

Proposition. The complete bipartite graph $K_{3,3}$ does not embed in the sphere.

Theorem. (Kuratowski's Theorem) A graph can be embedded in the plane (or S^2) if and only if it has no subgraph that is a *subdivision of* ...

Question. What can we figure out about embedding graphs in other surfaces besides S^2?

Proposition. If a graph G can be embedded in an orientable surface of genus k (without boundary), then it can be embedded in every orientable surface of higher genus.

Definition. The *genus* $\gamma(G)$ of a graph G is the minimal genus of an orientable surface (without boundary) in which G embeds.

Definition. A graph G is said to be *minimally embedded* if ...

Proposition. If a graph G is minimally embedded in a surface M, then the regions of $M - G$ are all topological disks.

Note. If a graph G is minimally embedded in a surface M, then $v - e + f = \chi(M)$, where v is the number of vertices of G, e is the number of edges of G, and f is the number of regions of $M - G$.

Embedding Complete Graphs

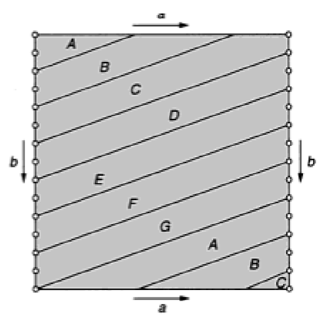
Question. If K_n is embedded in an orientable surface M without boundary, what can we say about $\chi(M)$?

Question. If K_n is embedded in an orientable surface M without boundary, what can we say about the genus of M?

Question. How does this relate to the coloring number for an orientable surface M?

Practice problems

1. True or False:
   (a) Any map in the plane can be colored with at most six colors.
   (b) The coloring number of S^2 is 4.
   (c) Seven colors are necessary to e prepared to color any possible map on T
   (d) There may be a map on S^2 requiring six colors.
   (e) Any map on $2T$ can be colored with at most eight colors.
2. (a) How many colors are needed to color the map on T shown in the figure below? Notice that all vertices are along side b.



   (b) Could you alter this map of T to require more colors? Explain briefly.
3. How many colors are needed to be confident that you could color all maps on $5T$? Explain.

4. (a) Find the map-coloring number for the disk. (For this exercise, you may assume the stated map-coloring number for any surface, even if we did not prove it.)
   (b) Find the map-coloring number for the Mobius band (same assumption as part (a)).
5. Find a map of the sphere so that each country borders only two other countries but the map requires three colors to color it.
6. Construct a map in the plane by drawing a continuous closed curve that crosses itself as many times as you want, but it is never tangent to itself and never traces over itself. Conjecture how many colors are needed to color any such map. Can you prove your conjecture? Do you think your conjecture is true on all surfaces?
7. (a) Show that any trivalent map on the sphere has a face with fewer than six sides.
   (b) Show that any trivalent map on the torus has a face with fewer than seven sides.
8. In the proof that any map on the sphere can be five-colored, where do you run into trouble if you try to use the same argument to show that any map on S^2 can be four-colored?
9. Why doesn't the proof that any map on the sphere can be five-colored extend to the torus or other surfaces?
10. How many vertices and how many edges does $K_{m,n}$ have?

11. A *triangulation* of a surface M is a decomposition of M into faces such that every face has exactly three edges and any two faces meet along a single edge, at a single vertex, or not at all, and no face meets itself.

    For a triangulation on a surface M, prove that

    (a) $3f = 2e$

    (b) $e = 3[v - \chi(M)]$

    (c) $v \geq \left\lceil \dfrac{7 + \sqrt{49 - 24\chi(M)}}{2} \right\rceil$ (yes, shades of Haewood's number). Hint: think of the vertices and edges as a graph. What is the maximum number of edges a graph with v vertices can have?

    (d) Any triangulation of the sphere has at least 4 vertices, 6 edges, and 4 faces. Why? Show that this minimum can be realized.

    (e) Any triangulation of P has at least how many vertices? Show this minimum can be realized.

    (f) Any triangularion of T has at least how many vertices? Show this minimum can be realized. (Note: K is again an exception here. The formula gives a minimum of seven vertices, but that cannot be realized.)

12. For each case below, decide if a graph, without loop or parallel edges but with the specified number of vertices, can be embedded in S^2. Draw an embedding or

show there is none.

(a) $v = 7$ and $e = 17$

(b) $v = 7$ and $e = 15$

13. If a graph with six vertices can be embedded in S^2, what is the maximum number of edges it can have? Prove your answer and illustrate the maximal case.

14. Show that any pair (v, e) with $3f \leq 2e$ and $v \geq 3$ can be realized by a graph embedded in S^2. Here are some steps to lead you to this result:

    (a) Show that $f = 2 - v + e$ and use the inequality $3f \leq 2e$ to get an upper bound for e.

    (b) Suppose e equals the upper bound obtained in part (a). Give an inductive argument to show that this case can be realized for any $v \geq 3$.

    (c) Explain why part (b) shows that any case where e is less than the upper bound can be realized.

15. Embed $K_{2,3}$ in the sphere (or plane). Can you embed $K_{2,n}$ in general?

16. Suppose by removing one edge of a graph we are able to embed it in the sphere. Then where can we embed the entire graph? Explain.

17. Embed $K_{3,3}$ in the torus T. Can you embed $K_{3,4}$ or $K_{4,4}$ in the torus? As a challenge, try $K_{3,6}$.

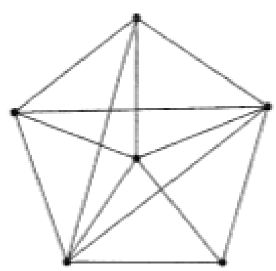
18. (a) Use $v - e + f = 0$ and $3f \leq 2e$ to show that K_8 cannot be embedded in T.
    (b) Draw a tesselation of the plane with equilateral triangles. Use this tesselation to find a nice embedding of K_7 in T. Hint: find a parallelogram in the tessellation that will serve as a plane model of T and, when the sides are identified to give T, the restricted graph is K_7.
19. Show that one cannot embed $K_4, 5$ in the torus.
20. Suppose a graph contains no simple closed curves (it's then called a tree). How many colors are needed to color it?
21. Suppose every vertex of a graph has valency 3 or less. Explain how to four-color the graph.
22. How many colors are needed to color $K_{m,n}$?
23. Show that the genus of K_7 is 1. dualize the embedding of K_7 in T from a previous exercise to create a map on T requiring seven colors.
24. Embed K_6 in K and use this embedding to find a map of K requiring exactly six colors.
25. Find a lower bound for the minimal genus of the *non-orientable* surface in which K_n embeds.
26. (a) Compute an upper bound for the coloring number for $4T$ using Haewood's

number H_M.

(b) How many colors are required to color the complete graph K_{12}? Explain briefly.

(c) Explain why parts (a) and (b) imply that K_{12} cannot be embedded in $4T$.

27. Determine on which orientable surface the graph with six vertices shown below can be minimally embedded. (Note that it is NOT a complete graph.)

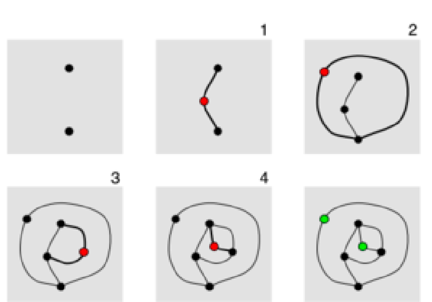


The Game of Sprouts

Begin with a surface M and some fixed number of vertices on M. There are two players. A player's move consists of drawing a path on M connecting two vertices and putting a new vertex in the middle of her path. No vertex is allowed to have more than three paths coming from it, and no path can cross itself or another path or go through a vertex. The winner is the last player to be able to make a move (draw a path).

Analyze the game on the sphere (or plane) first. Is there a winning strategy? Your answer may of course depend on the number of vertices you start with.

Explore what happens on other surfaces.



Another game, called Brussels Sprouts, starts with a number of crosses, i.e. spots with four free ends. A player moves by joining two free ends with a curve and then putting a short stroke across the line to create two new free ends. Again, a path cannot cross itself or another path or go through a vertex. The winner is the last player to be able to make a move.