

S27 Fractals

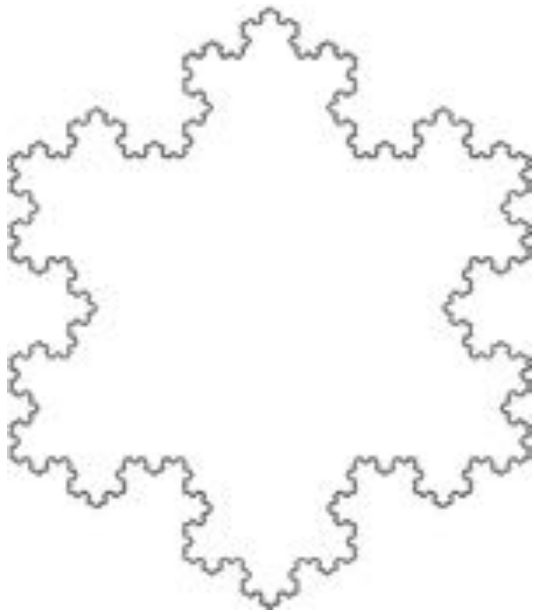
References:

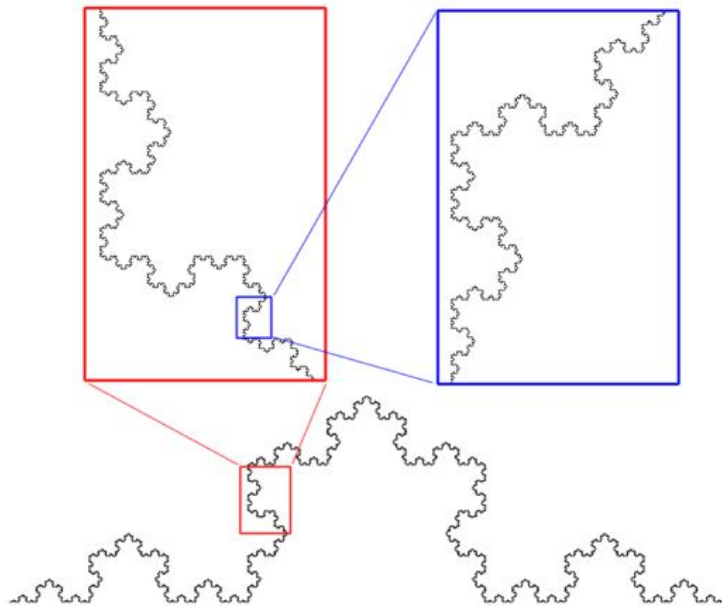
Supplies:

- Geogebra (phones, laptops)
- Business cards or index cards for building Sierpinski's tetrahedron or Menger's sponge.

What is a fractal?

Definition. A fractal is a shape that has





Animation at Wikimedia Commons

Question. Do fractals necessarily have reflection, rotation, translation, or glide reflection symmetry? What kind of symmetry do they have?

What natural objects approximate fractals?

Fractals in the world

A fractal is formed when pulling apart two glue-covered acrylic sheets.



High voltage breakdown within a block of acrylic creates a fractal Lichtenberg figure.



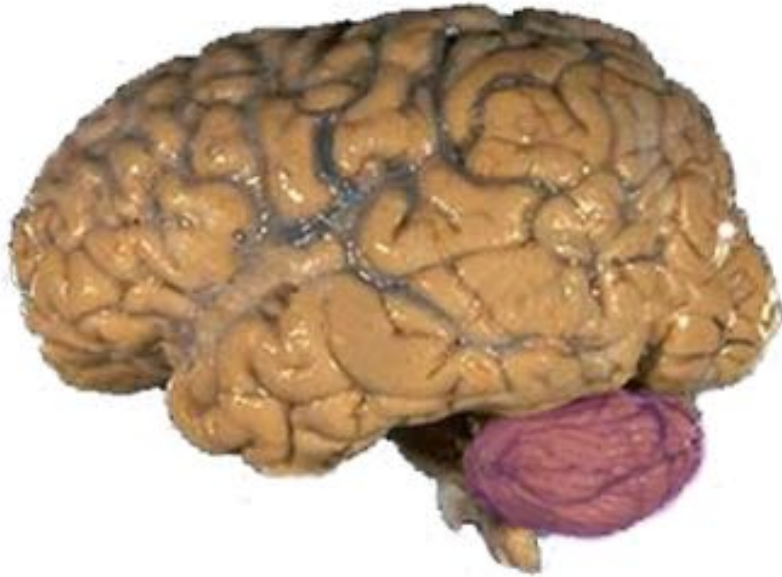
What happens to a CD in the microwave?

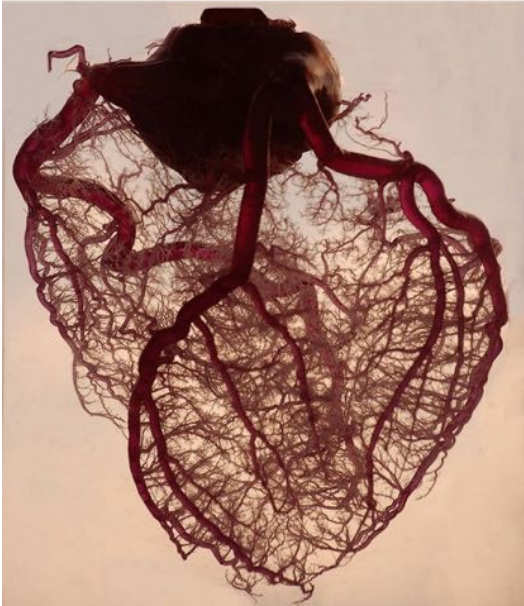


A DLA cluster grown from a copper sulfate solution in an electrodeposition cell.



Anatomy





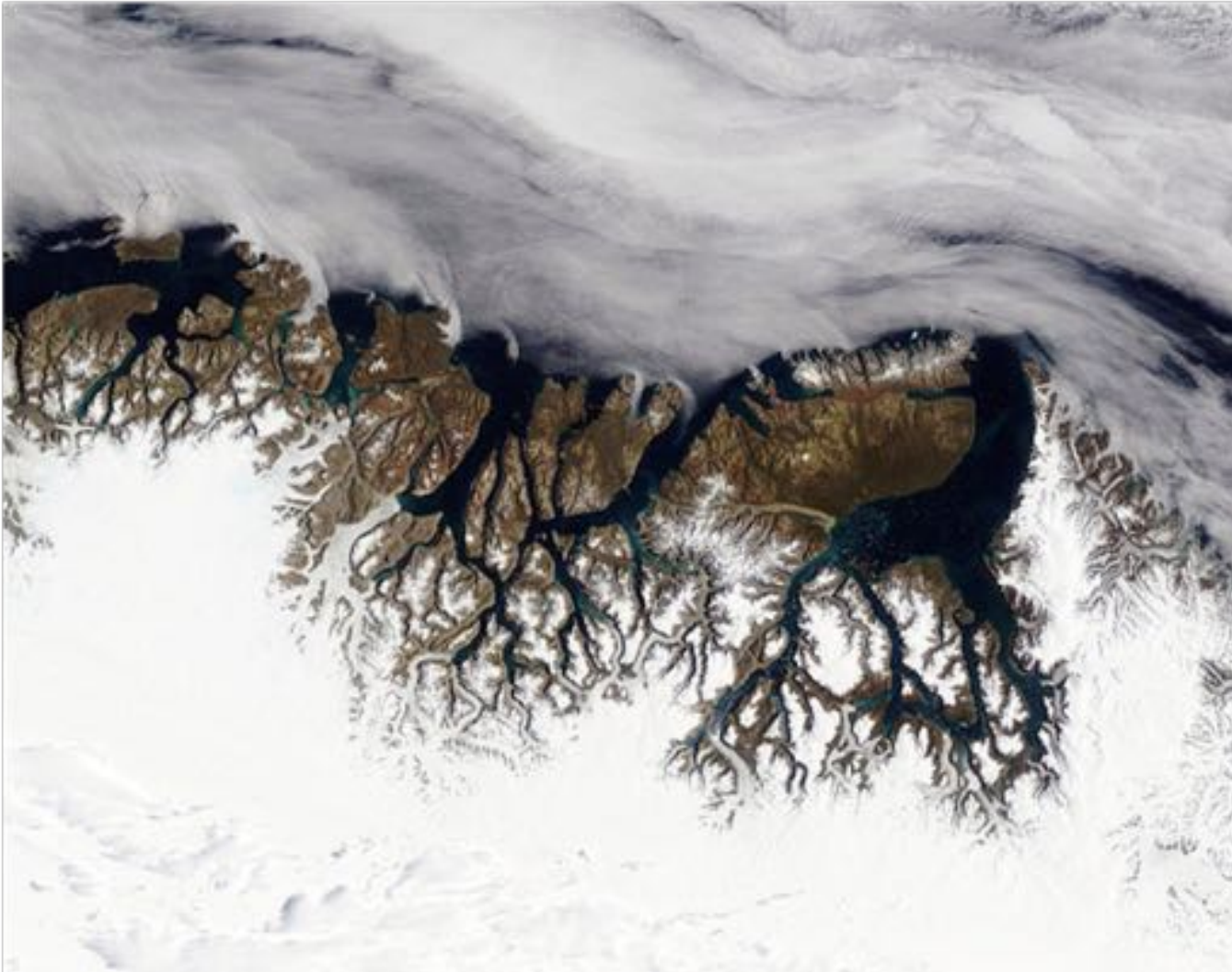
Natural tree



Simulated tree



Natural coastline



Simulated coastline



Natural mountain range



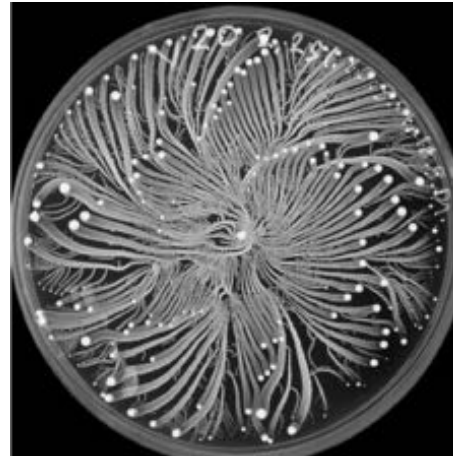
Simulated mountain range



The Great Wave off Kanagawa by Katsushika Hokusai



Patterns formed by bacteria grown in a petri dish.



Broccoli Romanesco



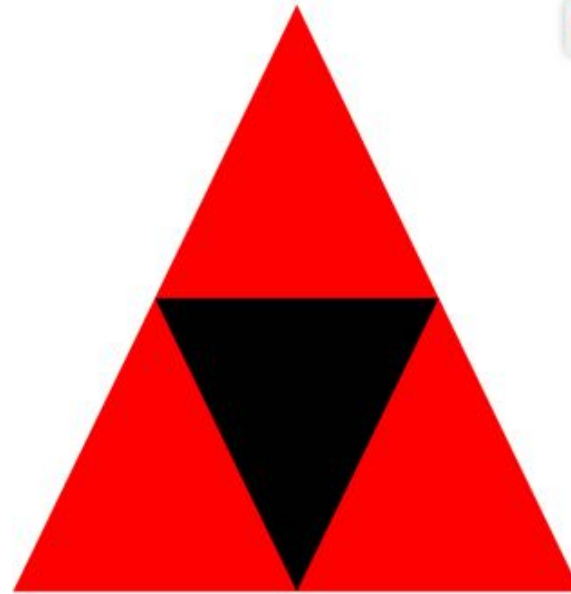
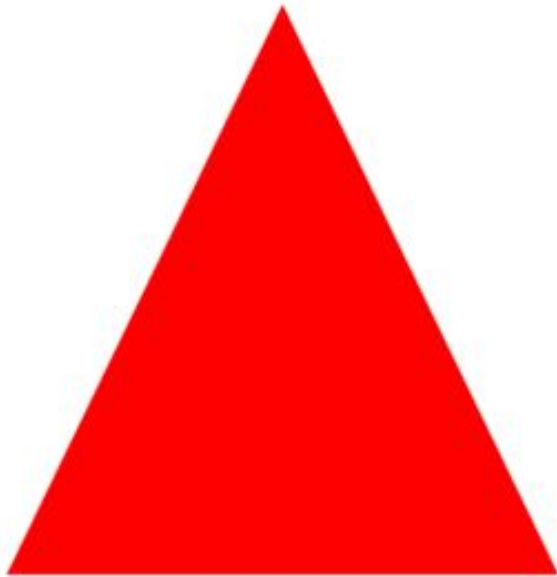
Fractal dimension

One way to think about dimension is to think about how many scaled copies of a figure fit into the figure.

- For a square
 - If we shrink the square by a factor of 2 (i.e. sides lengths are $1/2$ as long, then 4 copies of the square fit inside. $2^2 = 4$
 - If we shrink the square by a factor of 3 (i.e. sides lengths are $1/3$ as long, then 9 copies of the square fit inside. $3^2 = 9$
- For a cube
 - If we shrink the cube by a factor of 2 (i.e. sides lengths are $1/2$ as long, how many copies fit inside?
 - If we shrink the square by a factor of 3 (i.e. sides lengths are $1/3$ as long, how many copies fit inside?

Object	Shrinking factor	Number of copies	Equation	Dimension
Square	2	4	$2^d = 4$	$d = 2$
Square	3	9		
Cube	2			
Cube	3			

Sierpinski's tetrahedron



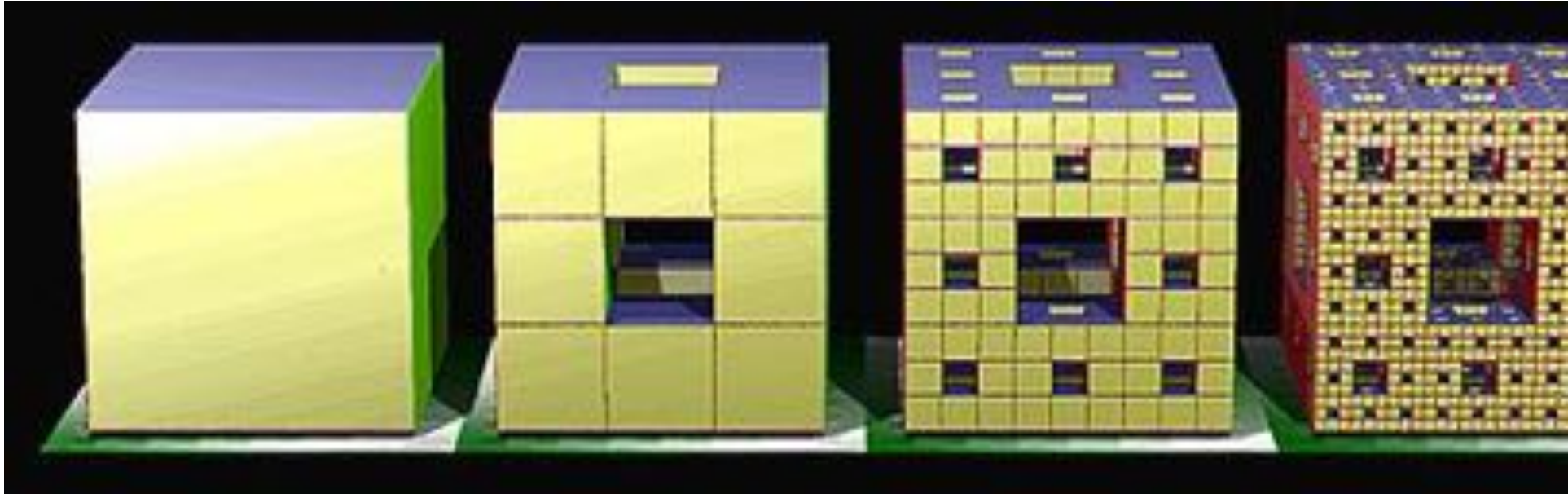
At each stage, remove the middle triangle from each visible triangle.

[Animation at Wikimedia Commons](#)

What is the dimension of the Sierpinski triangle?

Object	Shrinking factor	Number of copies	Equation	Dimension
Square	2	4	$2^d = 4$	$d = 2$
Cube	2	8	$2^d = 8$	$d = 3$
Sierpinski's Triangle	2	3		

Menger sponge



[Animation at Wikimedia Commons](#)

What is the fractal dimension of Menger's sponge?

Question. What is the general formula for the fractal dimension of any shape?

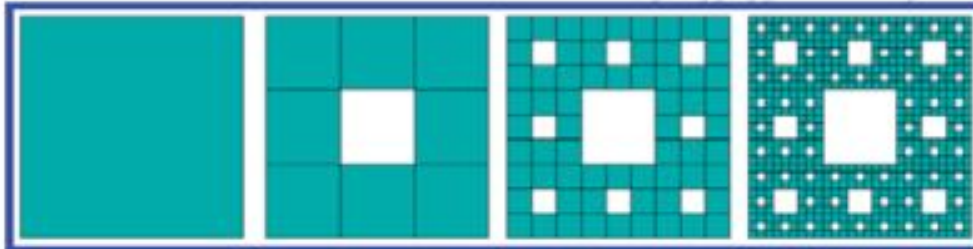
Question. How can you make a rough visual estimate of fractal dimension?

Question. Menger's sponge starts with a 3-dimensional cube, that has volume. What is an area analog of Menger's sponge? A length analog? What are the fractal dimensions of the resulting figures?

Question. Can you describe an analog that begins with a 4-dimensional hypercube?



Length analog, the Cantor Set



Area analog, the Sierpinski Carpet



Volume analog, the Menger Sponge

Question. How can you make analogs of Sierpinski's triangle in other dimensions? What are the fractal dimensions?

Topological dimension

There is another definition of dimension called topological dimension.

Definition. A set has topological dimension 0 if every point in the set has arbitrarily small neighborhoods whose boundaries are disjoint from A.

For example, a finite set of points in the plane is 0-dimensional.

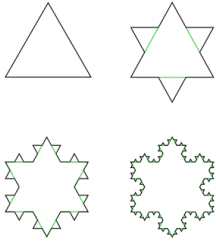
Definition. A set has topological dimension n if every point in the set has arbitrarily small neighborhoods whose boundaries meet the set in an $n - 1$ dimensional set and n is the smallest number for which that happens.

Example. . What is the topological dimension of a line? The surfaces of a sphere? The Cantor set? The Koch snowflake?

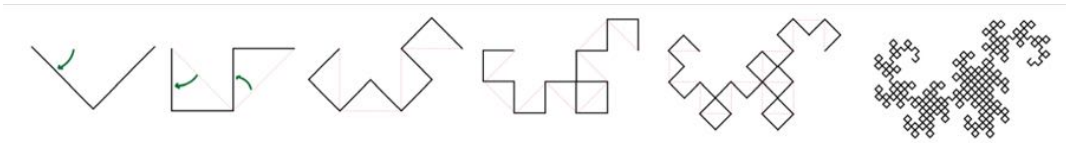
One definition of a fractal is a shape whose topological dimension is different from its fractal dimension. Is the topological dimension generally larger or smaller for the fractals we have looked at?

Practice problems

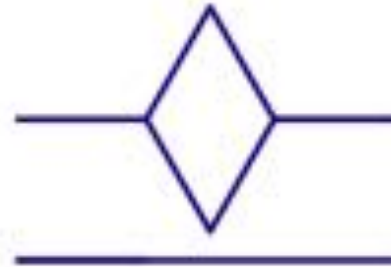
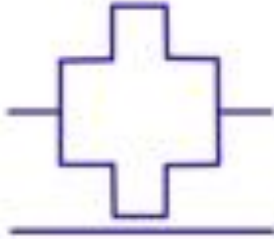
1. Calculate the fractal dimension of Koch's snowflake. Challenge: what is the total length of the boundary of the shape? What is the total area inside?



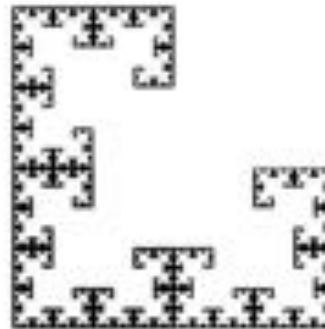
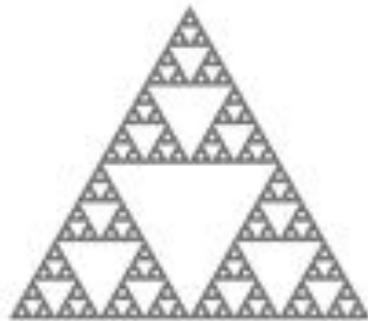
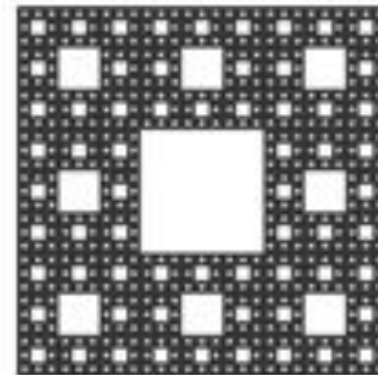
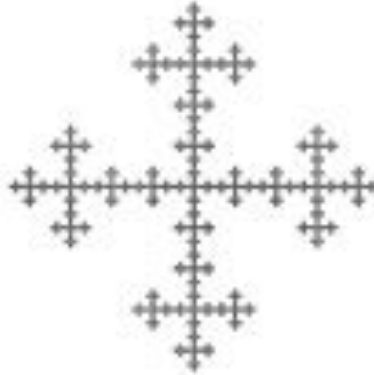
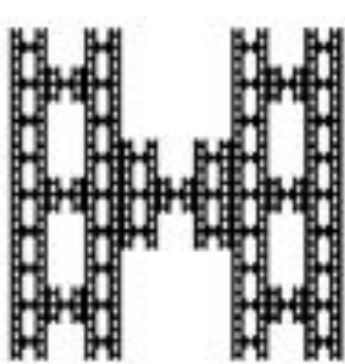
2. The dragon curve is generated by this replacement rule. Calculate its fractal dimension.



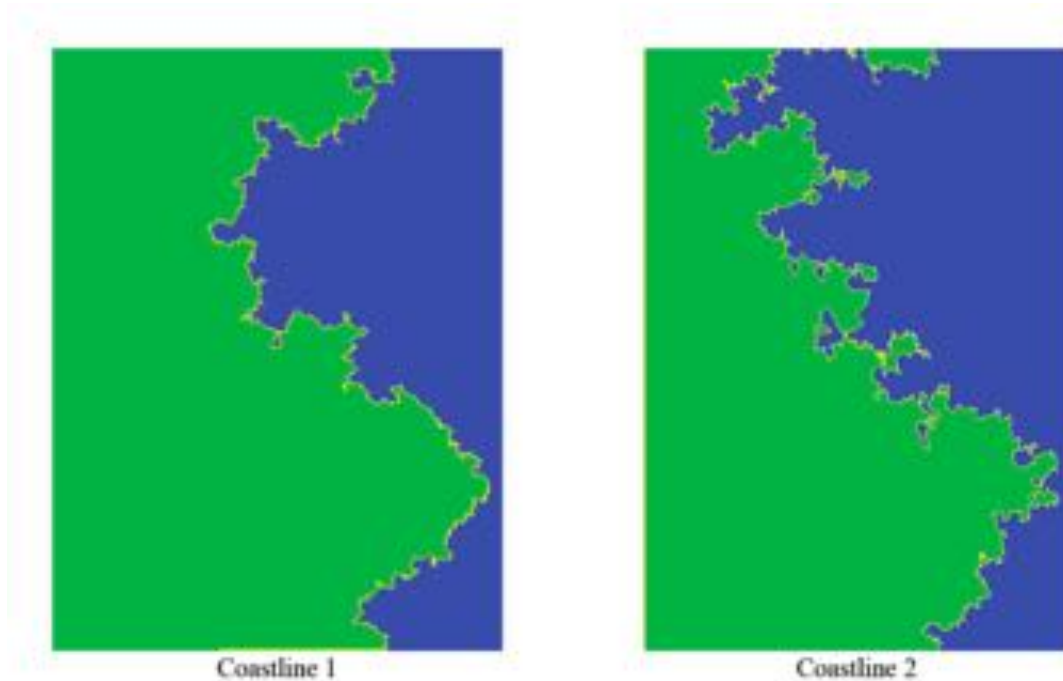
3. For each line fractal, the original segment is replaced by the fractal seed shown above the segment. Calculate the dimension of the resulting fractal and try to draw the fractal.



4. Calculate the fractal dimension of each fractal.



5. Without direct calculation, determine which coastline has the higher fractal dimension.



6. A certain fractal shape has fractal dimension 1.975. How many self-similar copies of the original seed shape will result from a scale factor of 5? Round answer to nearest whole number.
7. A fractal yields 8 copies using a scale factor of 2. How many copies will result using a scale factor of 16?

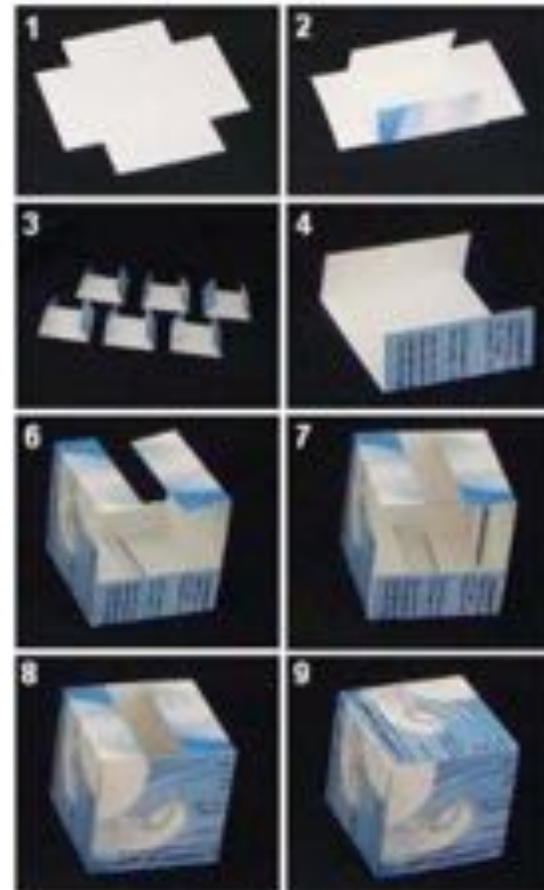
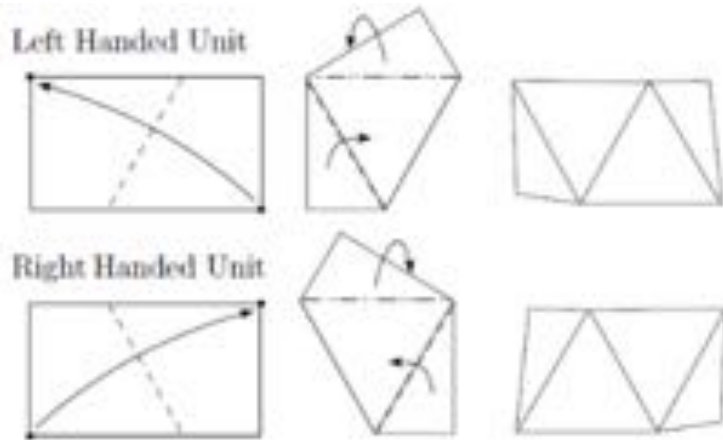
Vi Hart Fractal Videos

See the playlist of Vi Hart fractal videos.

Fractals: the Hidden Dimension Video

Building Sierpinski's tetrahedron and Menger's sponge

Use business cards folded into these units, and lots of tape, to build a Sierpinski tetrahedron or a Menger sponge.



Fractal applications

A fractal antenna uses a fractal, self-similar design to maximize the length of material that can receive or transmit electromagnetic radiation within a minimum total surface area or volume.



A really efficient cell phone antenna should have

- a very large boundary: more boundary means stronger reception
- a very small area: less area means more energy efficiency
- self-similarity: self-similarity means that the antenna works equally well at every frequency of the radio spectrum