§4 Classification of Surfaces

The goal of this section is to classify topological surfaces that are finite with no boundary.

References:

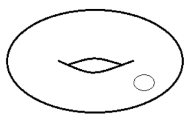
- *The Shape of Space* by Jeffrey Weeks, Chapter 5 and Appendix C

Supplies: None

Punctured Surfaces

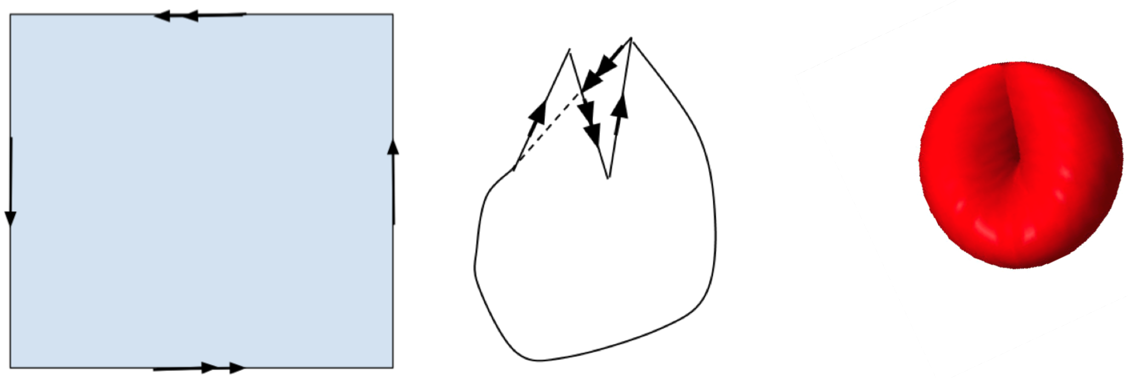
Definition. A closed *disk* is something topologically equivalent to the inside of a circle with its boundary included. An open disk is ...

Definition. A *punctured* surface is obtained by removing an open disk from the surface (leaving the boundary of the circle behind as part of the surface.

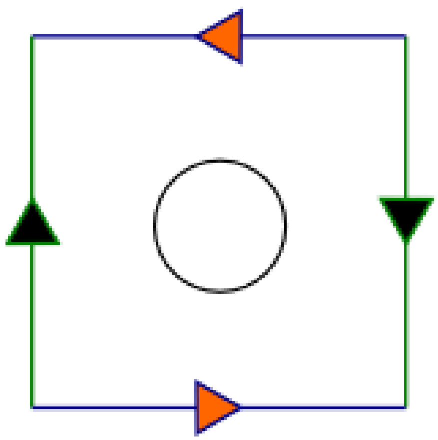


Question. What do you get when you puncture a sphere? When you puncture a disk? When you puncture a cylinder?

Question. What do you get when you puncture a projective plane?



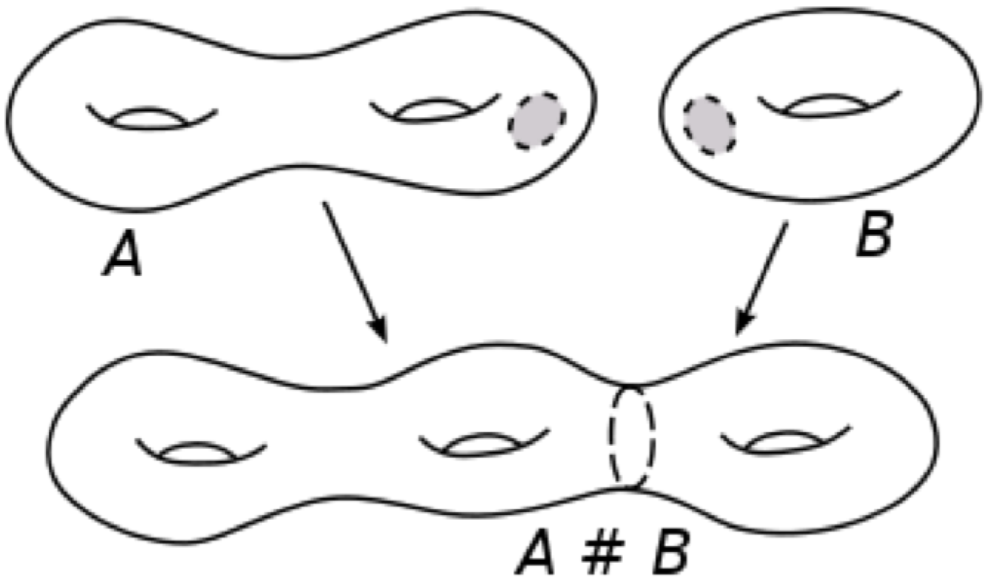
Hint: try cutting apart the surface, keeping track with arrows of how the gaping edges glue up, and rearrange things to get a gluing diagram that you recognize.



From http://www.math.cornell.edu/ mec/

Connected Sums

The connected sum of two surfaces is what you get when you puncture each surface and attach them together along the gaping boundaries.



Question. What do you get when you take the connected sum of a torus and a sphere? A Klein bottle and a sphere? In general, what happens when you connect sum with a sphere?

Question. What do you get when you take the connected sum of two tori?

Question. How is connected sum like addition for numbers? How is it different?

Question. What do you get when you take the connected sum of two disks?

A cylinder and a disk?

In general, what is the relationship between puncturing a surface and connected summing with a disk?

Question. What do you get when you take the connected sum of two projective planes?

To form the connected sum of two projective planes,

- First remove a disk from each of them.
- This gives us two ________________
- Now glue together the two boundary circles
- This gives us ________________

Building all surfaces

Write down 10 - 20 surfaces. Write down some infinite lists of surfaces. Try to include all possible compact surfaces on your list.

If we restrict ourselves to compact surfaces without boundary, it turns out (we will prove this later):

Theorem (Classification of Surfaces) Every closed surface is the connected sum of tori (T^2) and / or projective planes (P^2).

Question. If we want to think of a sphere as the connected sum of m tori and n projective planes, what should m and n be?

Number of Projective Planes

Number of Tori	0	1	2	3	4	
0						
1						
2						
3						
4						

Find a surface in the table that is equivalent to

- $K^2 \# P^2$
- $K^2 \# T^2$
- $K^2 \# K^2$

Are there any duplicates in this table?

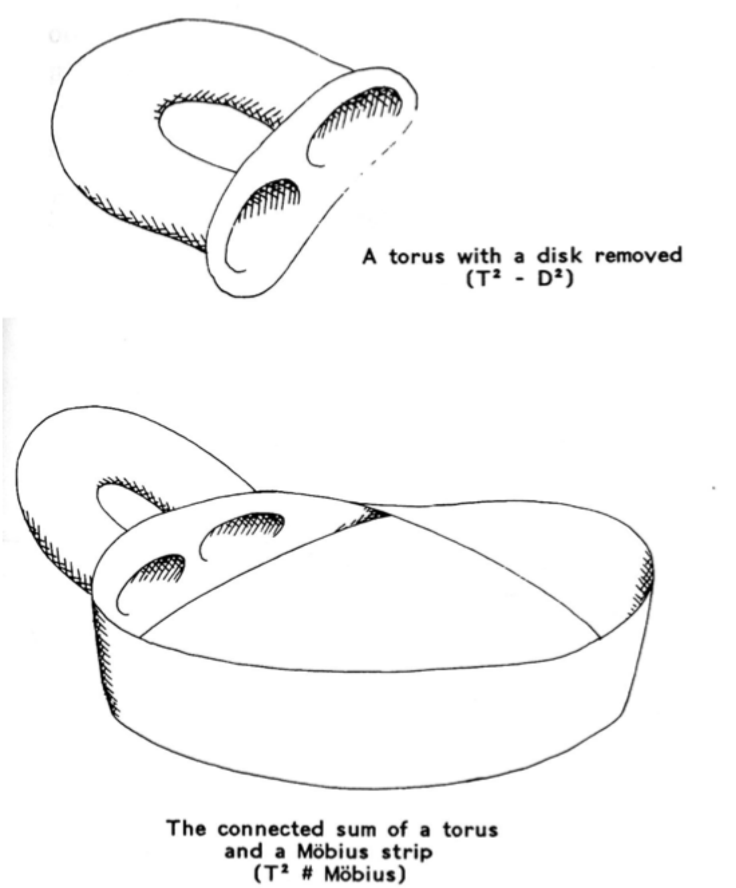
Handles, cross-handles, and cross-caps

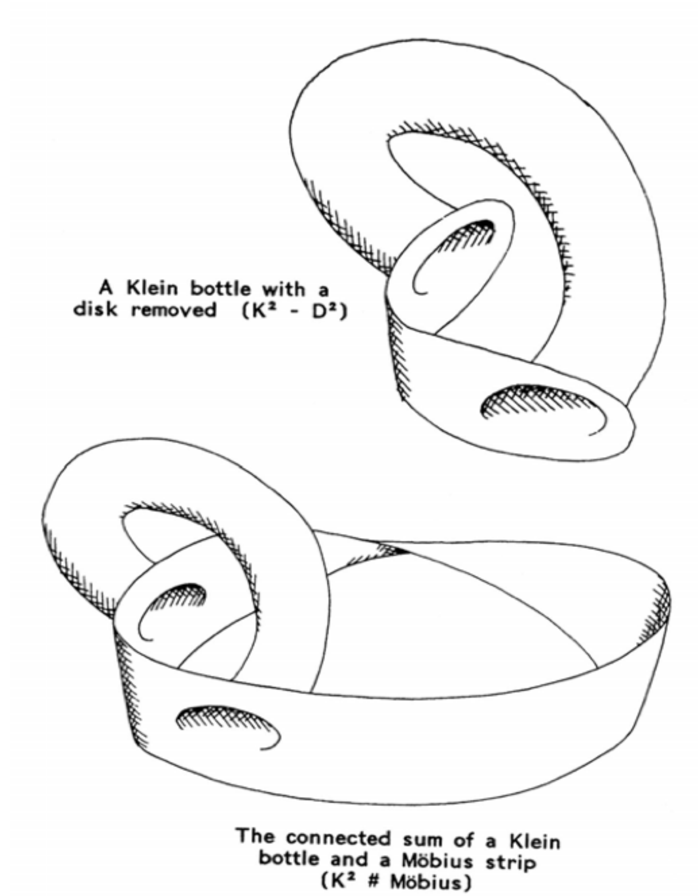
Definition. Connected summing with T^2 is called "adding a handle". Connected summing with K^2 is called "adding a cross handle". The surface P^2 is sometimes called a cross-cap.

There is an old saying: "Handles and crosshandles are equivalent in the presence of a crosscap. " Write an equation that conveys this same content. The pictures on the next page give clues on how to prove it.

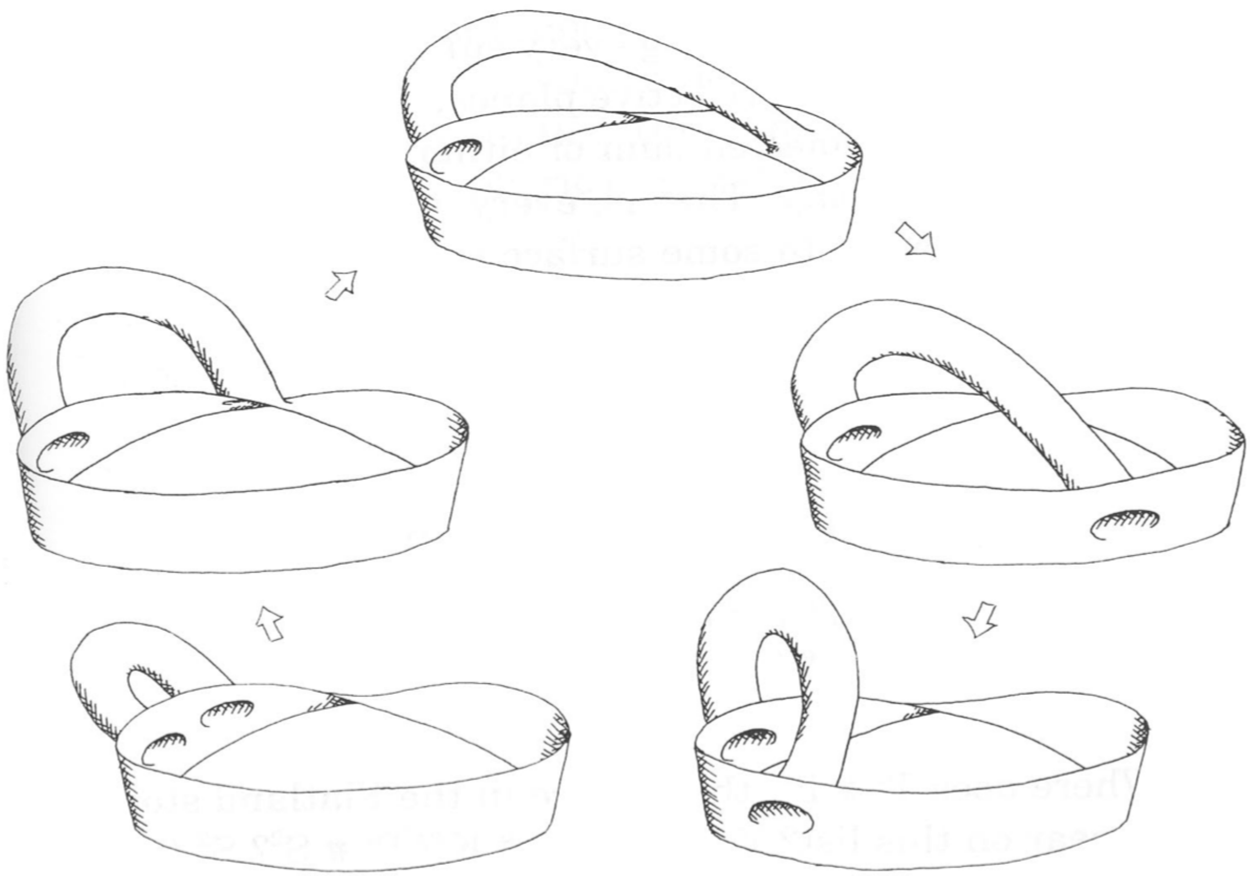
Connected sum with Mobius band

Study these pictures from Jeff Week's book *The Shape of Space*





Deforming a handle to a cross-handle



Question. How does this show that $T^2 \# P^2 \cong K^2 \# P^2$?

Question. Is it true that $T^2 \cong K^2$?

Question. Is it true that $T^2 \# K^2 \cong K^2 \# K^2$?

Reduced list of surfaces

Question. Fill in a reduced list of surfaces by eliminating known duplicates.

Number of Projective Planes

Number of Tori	0	1	2	3	4	
0						
1						
2						
3						
4						

Note. A complete list of closed surfaces can also be listed as follows.

$$S^2$$

T^2	P^2
$T^2\#T^2$	$P^2\#P^2$
$T^2\#T^2\#T^2$	$P^2\#P^2\#P^2$
$T^2\#T^2\#T^2\#T^2$	$P^2\#P^2\#P^2\#P^2$
$\dots$	$\dots$

Question. Are there any other duplicates in the list?

Match the surfaces

Match the surface from the left column with the equivalent surface from the right column.

Column A

$S^2 \# T^2$

K^2

$S^2 \# S^2 \# S^2$

$P^2 \# T^2$

$K^2 \# T^2 \# P^2$

Column B

$P^2 \# P^2$

$K^2 \# P^2$

$P^2 \# P^2 \# P^2 \# K^2$

$S^2 \# S^2$

T^2

From The Shape of Space by Jeff Weeks

If we now consider compact surfaces WITH boundary, the following theorem holds (we will prove it later)

Theorem. Any compact surface with boundary is topologically equivalent to one of the following surfaces:

Conway's ZIP Proof

Theorem. (Classification of Surfaces) Any compact surface without boundary is topologically equivalent to ....

and any compact, connected surface with boundary is topologically equivalent to ....

Note. It is enough to show that any compact, connected surface (with or without boundary) can be written as a connected sum of ...

We will call surfaces of this form "ordinary" and prove that all compact, connected surfaces are ordinary.

Proof:

Step 1: Any compact surface can be "triangulated": this means

Step 2:

- Triangulate the surface,
- Cut it apart into the triangles (whose sides are marked with "zips").
- Partially reassemble the triangles to get a gluing diagram for the surface: a disk with edges marked for further identification.

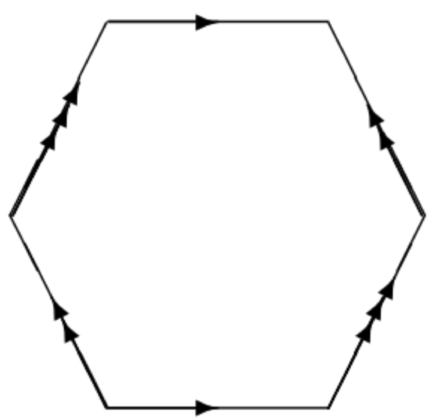
Step 3: Continue to zip up zip-pairs and check that each time we zip up a zip-pair in an ordinary surface, we still have an ordinary surface.

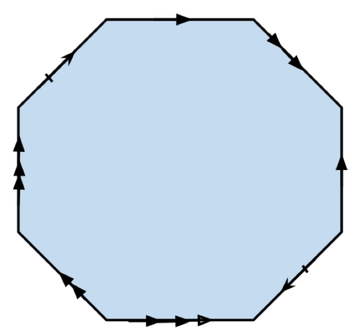
What are the possible ways that zip pairs could be situated on an ordinary surface?

For example, the two zips could be on separate boundary circles, and could both be the entire boundary circle. If that is the situation, how does our ordinary surface change when we zip the zip-pair?

Gluing diagram and surface classification problems

1. Draw a gluing diagram for $T\#T$. For clarity, you may want to label the edges of your polygon with letters instead of different kinds of arrows.
2. Construct a gluing diagram to represent nT.
3. • Is the connected sum of two orientable surfaces orientable or non-orientable?
   • What about the connected sum of two non-orientable surfaces?
   • What about an orientable surface and a non-orientable surface?
4. Which of these gluing diagrams represent non-orientable surfaces?





5. How can you tell from a gluing diagram whether a surface is orientable or nonorientable? Give a method that a 6-year-old could understand.

6. Draw a gluing diagram for the connected sum of m projective planes mP.
7. (a) If $(kT)\#mP) \equiv (nT)\#K$, show that m is even.

   (b) If $(kT)\#(mP) \equiv n(T\#P)$, express m in terms of k and n.
8. (Challenge) What is the maximal number of points on a sphere so that you can join each pair of points by a curve on the sphere without any two curves meeting each other (except at their ends)? What about on a torus? Experiment with some other surfaces and report on your conjectures.